Approximate groups and Hilbert's 5-th problem

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But it is known to hold if $\dim(M) = 2$ (Montgomery-Zippin 1940's) or 3 (J. Pardon 2012), and the general case is reduced to determining whether $\mathbb{Z}_p$ (the p-adic integers) can act faithfully on a manifold or not (Hilbert-Smith conjecture).

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Corollary (Montgomery-Zippin 1951): if G is locally compact and acts (faithfully) transitively on a manifold M, then G is a Lie group.

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"generalized Lie group" means:

- there is $G' \leqslant G$, open subgroup, s.t.
- $\forall U$ neighborhood of the identity in G', $\exists N \subset U$, with
- $N =$ closed normal subgroup N of G' s.t. G'/N is a Lie group with finitely many connected components.

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Totally disconnected case

Theorem (Van Dantzig, 1920's)

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and Van Dantzig's result we see that we may assume that G/G^0 is compact in proving the Gleason-Yamabe theorem.

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Proof strategy

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The original proof of this theorem was notoriously technical. It roughly goes as follows:

Definition: Say that G is **NSS** if it has "No Small Subgroups", i.e. $\exists U$ a neighborhood of the identity in G containing no non-trivial subgroups.

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- show that G is a generalized NSS group (i.e. replace "Lie group" by "NSS group" in the above statement)
- show that NSS groups are Lie groups.

Key to both steps are the so-called Gleason-Yamabe lemmas, whose purpose is to show that the property for an element $g \in G$ of being close to the identity together with all its powers up to some large number is closed under products.

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One way to phrase it is in terms of escape norms:

$$||g||_U := \inf\{\frac{1}{n+1}; g^i \in U \text{ for all i } \leqslant n\}.$$

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Lemma (Gleason-Yamabe lemmas)

If G is locally compact, then every neighborhood of the identity contains a smaller neighborhood U such that for $g, h \in U$

(i) $||hgh^{-1}||_U \leqslant C||g||_U$

(ii) $||gh||_U \leqslant C(||g||_U + ||h||_U)$

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From (i) and (ii), we get that $H := \{g \in U, ||g||_U = 0\}$ is a normal subgroup. And $\langle U \rangle / H$ is NSS.
From (ii) and (iii), one can derive that if $X(t)$ and $Y(t)$ are one-parameter subgroups $\mathbb{R} \mapsto G$, then

$$X + Y := t \mapsto \lim_n (X(\frac{1}{n})Y(\frac{1}{n}))^{[nt]}$$

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As a consequence:

Proposition

Let G be a locally compact group and $L(G)$ be the set of one-parameter subgroups. Then $L(G)$ has naturally the structure of a topological vector space.

Proof of the Gleason-Yamabe lemmas

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The only essential ingredients are :

(i) the existence of the Haar measure.

(ii) the Peter-Weyl theorem.

Proof of the Gleason-Yamabe lemmas

Key idea: compare the escape norm $||g||_U$ to the L^∞ norm of

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Observation: If say $\phi(1) \geqslant 1$ and ϕ is supported on U, then

$$||\partial_g \phi||_\infty < 1 \text{ implies } g \in U.$$

- One side is then easy: $||g||_U \leqslant ||\partial_g \phi||_\infty$.

Indeed: if $||\partial_g \phi||_\infty \leqslant \frac{1}{n}$, then $||\partial_{g^i} \phi||_\infty \leqslant \frac{i}{n} < 1$ if $i \leqslant n$, so $g^i \in U$ and $||g||_U \leqslant \frac{1}{n}$.

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• For the other side, i.e. $||\partial_g \phi||_\infty \leqslant O(||g||_U)$, one writes the following formal Taylor expansion:

$$\partial_{g^n}\phi = n\partial_g \phi + \sum_{i=1}^{n} \partial_{g^i}\partial_g \phi$$

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Key idea of Gleason: Choose ϕ of the form $\psi_1 * \psi_2$, so as to get bounds on 2nd derivatives from Lipschitz bounds on ψ_1 and ψ_2.

Recall our theorems from yesterday:

G = a group.
$A \subset G$ a finite subset.

Theorem (BGT 2011 weak form)

Assume $|AA| \leqslant K|A|$. Then there is a virtually nilpotent subgroup $\Gamma \leqslant G$ and $g \in G$ such that

$$|A \cap g\Gamma| \geqslant |A|/O_K(1).$$

Back to approximate groups and the main theorem

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Theorem (BGT strong form: structure of approximate groups)

Assume $A \subset G$ a finite K-approximate subgroup. Then

$$A \subset XP,$$

where

- $|X| \leqslant O_K(1)$,
- *P is a coset nilprogression of rank and step $O_K(1)$,*
- $P \subset A^4$.

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Pick G_n a sequence of groups, and $A_n \leqslant G_n$ any sequence of finite K-approximate subgroups. Then form the ultra-product:

$$\mathbb{A} := \prod_{\mathcal{U}} A_n$$

This is still a K-approximate group, albeit infinite. We call it an ultra-approximate group.

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This is still a K-approximate group, albeit infinite. We call it an ultra-approximate group.

There are 3 main steps in the proof of the main theorem.

Step 1, build a locally compact topology

This step is due to E. Hrushovski. It is based on the following lemma due to Croot-Sisask, Sanders and Hrushovski (independently):

Lemma (Square roots of approximate groups)

If A is a finite K-approximate group, then for every $k \geqslant 2$, there is $S \subset A^4$ with $1 \in S = S^{-1}$ such that $S^k \subset A^4$ and $|S| \geqslant |A|/O_{k,K}(1)$.

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Pick $S_{k,n} \leqslant A_n^4$ as above and set $\mathbb{S}_k := \prod_{\mathcal{U}} S_{k,n}$. This defines a base of neighborhoods for a locally compact topology on $\langle \mathbb{A} \rangle$.

The Gleason-Yamabe theorem then allows to define the *Hrushovski Lie group* L associated to $\mathbb{A}$. It is the (unique) local quotient of $\langle \mathbb{A} \rangle$ with no non-trivial compact subgroups.

Corollary (Hrushovski)

If L is trivial, then there are finite subgroups $H_n \leqslant A_n^4$ such that $|H_n|/|A_n|$ is bounded (above and below).

The Hrushovski Lie group

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If L is trivial, then there are finite subgroups $H_n \leqslant A_n^4$ such that $|H_n|/|A_n|$ is bounded (above and below).

This already allowed Hrushovski to prove a weak form of our theorem: e.g. he showed that A has a subset A' with $|A'| > |A|/O_K(1)$, which is stable under group commutators, i.e. $[A', A'] \subset A'$.

Step 2: adapt the Gleason-Yamabe lemmas

We prove a version of the Gleason-Yamabe lemmas for finite approximate groups. Again, the escape norm for any $U \subset A^4$ is defined as:

$$||g||_U := \inf\{\frac{1}{n+1}; g^i \in U \text{ for all i } \leqslant n\}.$$

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If A is a finite K-approximate group, then A^4 contains an approximate subgroup U with $|A| \leqslant O_K(1)|U|$ such that for $g, h \in U$

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Consequences of the G-Y lemmas for approximate groups

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- from (i) and (ii) we see that $H := \{g \in U; ||g||_U = 0\}$ is a subgroup normalized by U.
- from (ii) and (iii) we see that the element e with smallest non zero norm $||e||_U = \inf\{||g||_U \neq 0, g \in U\}$ is centralized by U modulo H.

Step 3: build the nilprogression and conclude

(1) Define $A'_n \leqslant A_n^4$ as the set U_n obtained in the Gleason-Yamabe lemmas, and quotient out $H_n \leqslant A'_n$ to obtain $A''_n := A'_n/H_n$. Then form the ultra-product.

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(3) Quotient it out and induct on $\dim L$.

At the end the Hrushovski Lie group L of any ultra-approximate group $\mathbb{A}$ is shown to be nilpotent and we have exhibited the coset nilprogression.

Dziękuję bardzo!