Poincaré inequalities, rigid groups and applications

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Affine actions

An affine map:

$$Av = Lv + b,$$

where L is a linear map and b is a fixed vector.

Given a representation π of a group G on a linear space V an affine π-action is of the form

$$A_g v = \pi_g v + b_g,$$

where $g \mapsto b_g$ is a map $b : G \to V$ such that

$$b_{gh} = \pi_g b_h + b_g.$$

A vector $v \in V$ is a fixed point of this action if and only if b is a coboundary:

$$b_g = v - \pi_g v$$

for every $g \in G$.

$H^1(G, \pi) =$ 1-cocycles/1-coboundaries

Property (T)

Property (T) was defined by Kazhdan in late 1960'ies.

We use a characterization of (T) due to Delorme – Guichardet as a definition.

Definition

A group G has Kazhdan's property (T) if every action of G by affine isometries on a Hilbert space has a fixed point.
Equivalently,

$$H^1(G, \pi) = 0$$

for every unitary representation π.

Property (T)

Property (T) is a powerful rigidity property and has many applications:

- construction of expanders – Margulis (1973). Finite quotients of a residually finite group with (T) form a family of expanders;
- solution of the Ruziewicz problem – Sullivan, Margulis (1980), based on work of J. Rosenblatt.
- rigidity for operator algebras and dynamical systems (Connes, Jones, Popa,...)
- ...

Proving (T) is always non-trivial.
Examples:
– $SL_n(\mathbb{Z})$ has (T) for $n \geq 3$,
– amenable and, more generally, a-T-menable groups do not have (T).

Generalizing (T) to other Banach spaces

X – reflexive Banach space

We are interested in groups G for which the following property holds:

every affine isometric action of G on X has a fixed point

or equivalently,

$$H^1(G, \pi) = 0$$

for every isometric representation π of G on X.

This is usually much more difficult than for the Hilbert space, even when $X = L_p$.

Previous results

Only a few positive results are known:

- (T) $\Longleftrightarrow$ fixed points on L_p and any subspace, $1 < p \leq 2$
- (T) $\Longrightarrow\ \exists\, \varepsilon = \varepsilon(G)$ such that fixed points always exists on L_p for $p \in [2, 2+\varepsilon)$ (Gromov, Fisher–Margulis)
  (a general argument, ε unknown)
- lattices in higher rank Lie groups for $X = L_p$ for all $p > 1$ (Bader – Furman –Gelander – Monod, 2007)
- $SL_n(\mathbb{Z}[x_1, \ldots x_k])$ for $n \geq 4$; $X = L_p$ for all $p > 1$ (Mimura, 2010)
- Gromov monsters for $X = L_p$ for all $p > 1$ (Naor – Silberman, 2010)

Some of these results also extend to Shatten p-class operators (Puschnigg) and other non-commutative L_p-spaces (B. Olivier)

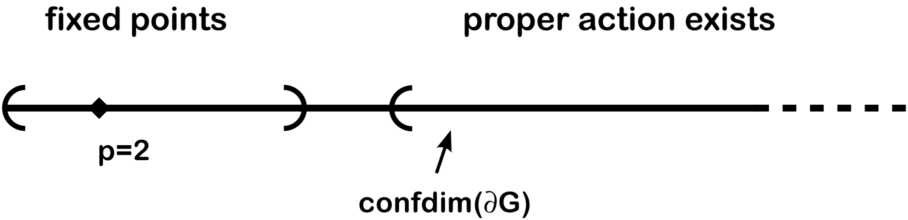
Previous results

Some groups with property (T) admit fixed point free actions on certain L_p.

- $G = Sp(n, 1)$ admits fixed point free actions on $L_p(G)$, $p \geq 4n + 2$ (Pansu 1995)
- hyperbolic groups admit fixed point free actions on $\ell_p(G)$ for $p \geq \text{confdim}(\partial G)$ (Bourdon and Pajot, 2003)
- for every hyperbolic group G there is a $p > 2$ (sufficiently large) such that G admits a metrically proper action by affine isometries on $\ell_p(G \times G)$ (Yu, 2006)
- proper actions exist for hyperbolic G and action on $L_p(\partial G \times \partial G)$ for p sufficiently large (Nica, 2012)

Values of p (after Cornelia Drutu)

Consider e.g. a hyperbolic group G with property (T).



There are many natural questions about the above values of p.

Let $\mathscr{P} = \left\{ p : H^1(G,\pi) = 0 \text{ for every isometric rep. } \pi \text{ on } L_p \right\}$

The only thing we know about $\mathscr{P}$ is that it is open.

Question: Is $\mathscr{P}$ connected?

Spectral conditions for property (T)

Based on the work of Garland,

used to prove (T) by Pansu, Żuk, Ballmann – Świątkowski, Dymara – Januszkiewicz

Theorem (general form)

Let G be acting properly discontinuously and cocompactly on a 2-dimensional contractible simplicial complex K and denote by $\lambda_1(x)$ the smallest positive eigenvalue of the discrete Laplacian on the link of a vertex $x \in K$. If

$$\lambda_1(x) > \frac{1}{2}$$

for every vertex $x \in K$ then G has property (T).

Other implications

Under roughly the same assumptions fixed points always exist for isometric actions on:

- complete metric spaces of non-positive curvature (M.-T. Wang, 2000)
- certain complete metric $CAT(0)$ metric spaces (H. Izeki, S. Nayatani, 2005)

This approach uses harmonic maps to prove existence of fixed points.

Link graphs on generating sets

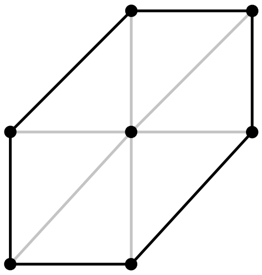
G - group, $S = S^{-1}$ - finite generating set of G, $e \notin S$.

Definition

The link graph $\mathcal{L}(S) = (V, E)$ of S:

- vertices $V = S$,
- $(s, t) \in S \times S$ is an edge $\in E$ if $s^{-1}t \in S$.

Example. Let $\mathbb{Z}^2$ be generated by $(1, 0)$, $(0, 1)$, $(1, 1)$ and their inverses.



The spectral criterion

Laplacian on $\ell_2(S, \deg)$:

$$\Delta f(s) = f(s) - \frac{1}{\deg(s)} \sum_{t \sim s} f(t)$$

λ_1 denotes the smallest positive eigenvalue

Theorem (Żuk, 2003)

If $\mathcal{L}(S)$ connected and $\lambda_1(\mathcal{L}(S)) > \dfrac{1}{2}$ then G has property (T).

Poincaré inequalities

Let $Mf = \sum_{x \in V} f(x) \dfrac{\deg(x)}{\#E}$ be the mean value of f

Definition (p-Poincaré inequality for the norm of X)

X-Banach space, $p \geq 1$, $\Gamma = (V, E)$ - finite graph. For every $f : V \to X$

$$\left(\sum_{s \in V} \|f(s) - Mf\|_X^p \deg(s) \right)^{1/p} \leq \kappa \left(\sum_{(s,t) \in E} \|f(s) - f(t)\|_X^p \right)^{1/p}.$$

The inf of κ for $\mathcal{L}(S)$, giving the optimal constant, is denoted $\kappa(p, S, X)$

The classical p-Poincaré inequality when $X = \mathbb{R}$.

1. $\kappa(1, S, \mathbb{R}) \simeq$ Cheeger isoperimetric const
2. $\kappa(2, S, \mathbb{R}) = \sqrt{\lambda_1^{-1}}$;
3. for $1 \leq p < \infty$ we have $\kappa(p, S, L_p) = \kappa(p, S, \mathbb{R})$

The Main Theorem

Given $p > 1$ denote by p^* the adjoint index: $\frac{1}{p} + \frac{1}{p^*} = 1$.

Main Theorem

Let X be a reflexive Banach space, G a group generated by S as earlier. If $\mathcal{L}(S)$ is connected for some $p > 1$

$$2^{-\frac{1}{p}}\kappa(p, S, X) < 1 \qquad \textit{and} \qquad 2^{-\frac{1}{p^*}}\kappa(p^*, S, X^*) < 1,$$

then

$$H^1(G, \pi) = 0$$

for any isometric representation π of G on X.

Remark 1. By reflexivity, the same conclusion holds for actions on X^*

Remark 2. The roles of the two constants in the proof are different.

The Main Theorem

Remark 3. In the case

$$p = 2 \qquad \text{and} \qquad X = \text{Hilbert space}$$

this reduces to the spectral criterion for property (T) in the form given by Żuk.

Then $p = p^*$ so

$$\kappa(p^*, S, X^*) = \kappa(p, S, X)$$

and

$$2^{-1/p}\kappa < 1 \qquad \Leftrightarrow \qquad \lambda_1 > \frac{1}{2}.$$

Remark 4. It is possible to give a similar criterion using harmonic maps for many metric spaces, including e.g., metric spaces.

However, that criterion is in terms of the first eigenvalue of the p-Laplacian:

$$\lambda_1^{(p)} = \frac{\|\nabla f\|_X}{\min_{c\in\mathbb{R}} \|f - c\|_X} \qquad \text{versus} \qquad \kappa_p^{-p} = \frac{\|\nabla f\|_X}{\|f - Mf\|_X}.$$

The first constant is virtually impossible to compute, while the second allows for interpolation in the setting of, for example, L_p spaces.

Sketch of proof

Difficulty: lack of self-duality when X is not a Hilbert space

For any Hilbert space $\mathcal{H}^* = \mathcal{H}$, every subspace has an orthogonal complement

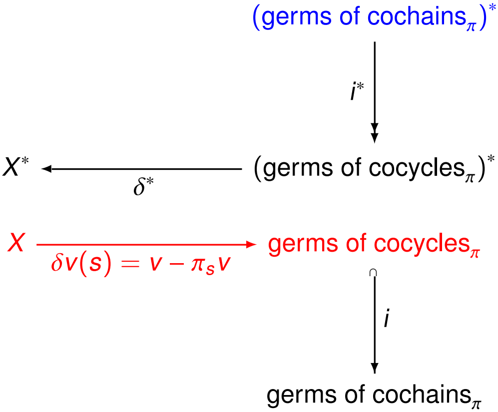
For $Y \subseteq X$ Banach spaces, Y might not have a complement,

$$Y^* = X^* / \operatorname{Ann}(Y)$$

[Ann(Y)=all functionals in X^* which identically vanish on Y]
with the quotient norm

$$\left\| \, [y] \, \right\|_{Y^*} = \inf_{x \in \operatorname{Ann}(Y)} \|y - x\|_{Y^*}$$

General remark: computing quotients of Banach spaces can be difficult: for instance, every separable Banach space is a quotient of $\ell_1(\mathbb{N})$.



X^* is equipped with the adjoint representation,

$$\overline{\pi}_g = \pi^*_{g^{-1}}.$$

We want to show that δ is onto.

This is equivalent to δ^* being bounded below

All calculations are restricted to the generators $\Rightarrow$ “germs” terminology

Step 1: identify (germs of cochains$_\pi$)*.

Theorem

If X-reflexive, π – isometric representation. Then

$$(\textit{g.o. cochains}_{\pi})^* \textit{ isometrically isomorphic to } (\textit{g.o. cochains}_{\overline{\pi}}).$$

Sketch of proof: we view g.o. cochains$_{\pi}$ as a complemented subspace of a larger Banach space, $\mathcal{Y}$:

$$\text{g.o. cochains}_{\pi} \oplus \mathcal{Z} = \mathcal{Y},$$

$$\text{g.o. cochains}_{\overline{\pi}} \oplus \overline{\mathcal{Z}} = \mathcal{Y}^*.$$

Moreover, g.o. cochains$_{\pi}$ is a 1-eigenspace of an involution Q_{π} on $\mathcal{Y}$.
We obtain

$$(g.o.cochains_{\pi})^* = \mathcal{Y}^*/\overline{\mathcal{Z}} \quad \text{isomorphic to} \quad \text{g.o. cochains}_{\overline{\pi}}$$

This is not sufficient – we need an isometric isomorphism.

Orthogonality-type conditions

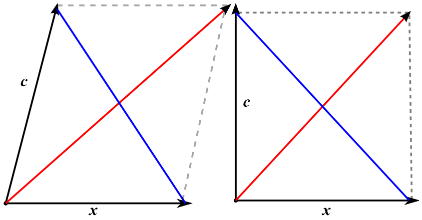
Proposition

If π is isometric then

$$\|c - x\|_{\mathcal{Y}} = \|c + x\|_{\mathcal{Y}},$$

for $c \in$ g.o. cochains$_{\overline{\pi}}$, $x \in \overline{\mathcal{Z}}$

Equivalently, if π is isometric then the involution Q_π is isometric as well. This is an orthogonality-type condition



This allows to compute $\delta^* i^*$

Step 2.

Proposition

$\delta^* i^* = M$, the mean value operator.

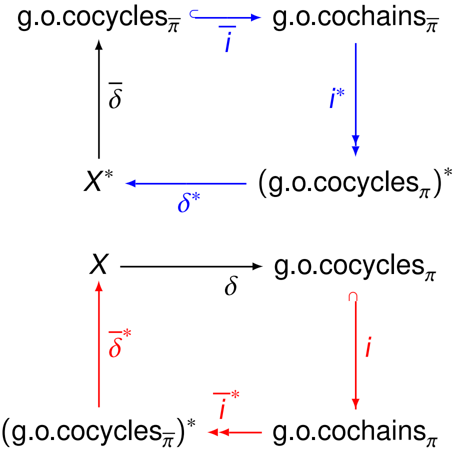
The next step would naturally be to identify (g.o. $\text{cocycles}_\pi)^*$, but it is not clear how to do it, except for some cases.

E.g., when X is isomorphic to the Hilbert space then we know that every subspace is complemented and then

$$(\text{g.o. cocycles}_\pi)^* \simeq \text{g.o. cocycles}_{\overline{\pi}}.$$

Question

What is the relation between $(\text{g.o. cocycles}_\pi)^*$ and g.o. $\text{cocycles}_{\overline{\pi}}$?



Thm 1. If $2^{1/p^*}\kappa(p^*, S, X^*) < 1$ then $\delta^* \, i^* \, \bar{i}$ is bounded below.

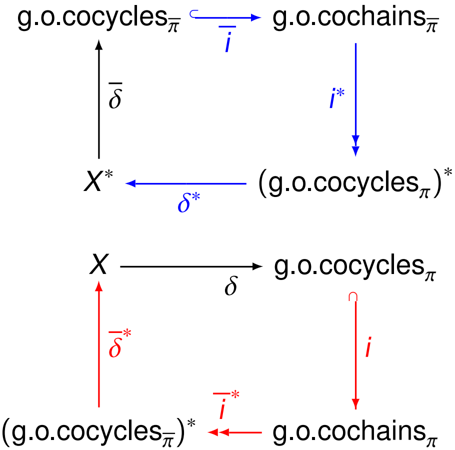
Thm 1 follows from a sequence of inequalities

It implies δ^* is bounded below on image of $i^* \, \bar{i}$

The same argument for the other inequality gives: $2^{1/p}\kappa(p, S, X) < 1$ then $\bar{\delta}^* \, \bar{i}^* \, i$ is bounded below

$\Rightarrow$ $\bar{i}^* \, i$ is bounded below

$\Rightarrow$ $i^* \, \bar{i}$ is surjective. □



Note that the two spectral conditions together imply that

(g.o.cochains$_\pi$)* and g.o.cochains$_{\bar\pi}$

are isomorphic via $i^*\bar i$

Examples and Applications

Isometric representations on L_p

We want to apply this to $X = L_p$, $p > 2$

Desired outcome: vanishing of cohomology for all L_p,

$$p \in [2, 2 + c),$$

where we can say something about c.

This cannot be improved and we cannot expect vanishing for all $2 < p < \infty$:

1. the assumptions of the theorem are almost never satisfied for p sufficiently large
2. the main theorem applies to certain hyperbolic groups

Estimating p-Poincaré constants is a hard analytic problem for $p \neq 1, 2, \infty$.

$\widetilde{A}_2$ groups

Cartwright, Młotkowski and Steger defined finitely presented groups G_q where $q = k^n$ for k - prime such that

$$\mathcal{L}(S) = \text{ incidence graph of a projective plane over a finite field.}$$

In the 60ies Feit and Higman computed spectra of such incidence graphs:

$$2^{-\frac{1}{2}}\kappa(2, S, \mathbb{R}) = \sqrt{\left(1 - \frac{\sqrt{q}}{q+1}\right)^{-1}} \quad \longrightarrow \quad \frac{1}{\sqrt{2}}.$$

We now want to estimate $\kappa(p, S, L_p)$ for these graphs.

Estimating the p-Poincaré constant

When $p \geq 2$, in finite dimensional spaces: $\|f\|_{\ell_p^n} \leq \|f\|_{\ell_2^n} \leq n^{1/2-1/p}\|f\|_{\ell_p^n}$.

- $\#V = 2(q^2+q+1)$,
- $\#E = 2(q^2+q+1)(q+1)$
- $\deg(s) = q+1$ for every $s \in S$

Similarly for $p^* < 2$.

Theorem

For each q=power of a prime we have

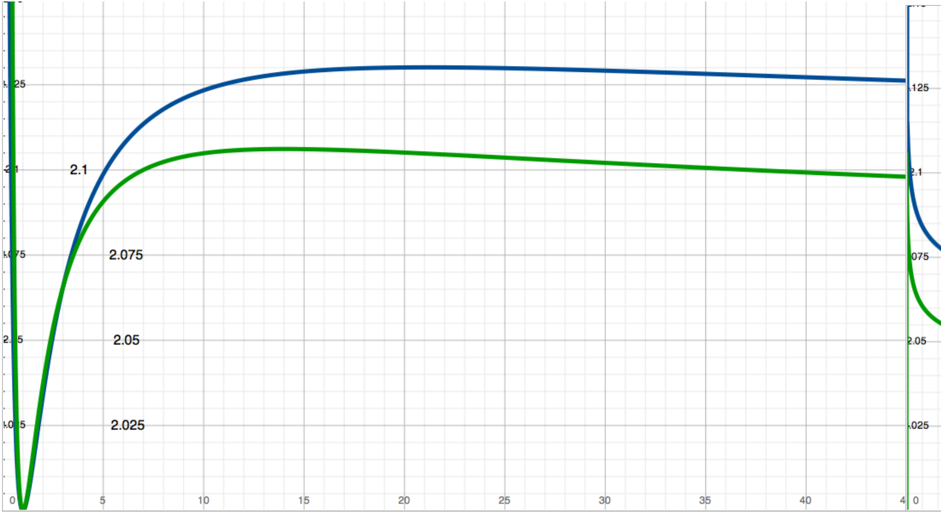
$$H^1(G_q,\pi) = 0$$

for any isometric representation π of G_q on any L_p for all

$$2 \leq p < \frac{2\ln\left(2(q^2+q+1)\right)}{\ln\left(2(q^2+q+1)\right) - \ln\sqrt{2\left(1-\frac{\sqrt{q}}{q+1}\right)}}.$$

Numerical values of p

We have $2 \leq p \leq 2.106$ and $p \to 2$ as $q \to \infty$.



Hyperbolic groups

Theorem (Żuk)

G in the density model for $1/3 < d < 1/2$ *is, with high probability, of the form*

$$H \twoheadrightarrow \Gamma \quad \subseteq_{f.i.} \quad G,$$

where G is hyperbolic and H has a link graph with $2^{-1/2}\kappa(2, S, \mathbb{R}) < 1$.

Detailed proof by M.Kotowski-M.Kotowski.
Vanishing of cohomology for *all* isometric representations on L_p is inherited by quotients and finite index subgroups.

Theorem

Let G, Γ *and* ϕ *be as above, and let* $\mathcal{L}(S) = (\mathcal{V}, \mathcal{E})$ *denote the link graph of* Γ. *Then* $H^1(G, \pi) = 0$ *for every isometric representation* π *of G on* L_p *for*

$$2 \leq p < \min\left\{p_0, \overline{p}_0{}^*\right\},$$

where

$$p_0 = \frac{\ln \deg_\omega - \ln(2\#\mathcal{E})}{\frac{1}{2}\ln\left(\frac{\deg_\omega}{\#\mathcal{E}}\right) - \ln \kappa(2, S, \mathbb{R})} \quad \textit{and} \quad \overline{p}_0 = \frac{\ln(\#\mathcal{V} \deg_\omega) - \ln 2}{\frac{1}{2}\ln(\#\mathcal{V} \deg_\omega) - \ln \kappa(2, S, \mathbb{R})}.$$

1. Conformal dimension

Definition (Pansu)

G hyperbolic, d_V -any visual metric on ∂G.

$$\mathrm{confdim}(\partial G) = \inf\left\{\dim_{Haus}(\partial G, d) : d \text{ quasi-conformally equiv. to } d_V\right\}.$$

$\mathrm{confdim}(\partial G)$ is a quasi-isometry invariant of G.

Difficult to compute or even estimate.

Problem (Gromov): Estimate the conformal dimension of random hyperbolic groups.

Theorem (Bourdon-Pajot, 2003)

G hyperbolic acts without fixed points on $\ell_p(G)$ for $p \geq \mathrm{confdim}(\partial G)$

1. Conformal dimension

We have the following estimate of the conformal dimension of ∂G

Theorem

For a random hyperbolic group as before ($1/3 < d < 1/2$),

$$\operatorname{confdim}(\partial G) \geq \min\left\{p_0, \overline{p}_0{}^*\right\},$$

where

$$p_0 = \frac{\ln \deg_\omega - \ln(2\#\mathcal{E})}{\frac{1}{2}\ln\left(\frac{\deg_\omega}{\#\mathcal{E}}\right) - \ln\kappa(2, S, \mathbb{R})} \quad \text{and} \quad \overline{p}_0 = \frac{\ln(\#\mathcal{V}\deg_\omega) - \ln 2}{\frac{1}{2}\ln(\#\mathcal{V}\deg_\omega) - \ln\kappa(2, S, \mathbb{R})}.$$

For $d < 1/16$ estimates of $\operatorname{confdim}(\partial G)$ were given by John Mackay.
Those groups are a-T-menable by a theorem of Haglund and Wise.

2. Shalom's conjecture

Let π be a *uniformly bounded* representation on a Hilbert space: a homomorphism $\pi : G \to B(H)$, such that

$$||\pi|| = \sup_{g \in G} ||\pi_g|| < \infty$$

General question: given a group G with property (T), does $H^1(G, \pi)$ vanish also for uniformly bounded representations π?

Yes for lattices in higher rank Lie gps (Bader-Furman-Gelander- Monod)
No for $Sp(n, 1)$ (Shalom, unpublished)

Conjecture (Shalom)

For every hyperbolic G there is a uniformly bounded π with $H^1(G, \pi) \neq 0$.

2. Shalom's conjecture

Let π be uniformly bounded on H. Change the norm on H to

$$\|v\|' = \sup_{g \in G} \|\pi_g v\|.$$

Let the new Banach space E be H with $\|\cdot\|'$.

Then π is an isometric representation on E, which is reflexive.

Additionally, the Poincaré constants satisfy

$$\kappa(2, S, E) \leq \|\pi\| \, \kappa(2, S, \mathbb{R}).$$

Theorem

For hyperbolic groups in the density model ($1/3 < d < 1/2$),

$$H^1(G, \pi) = 0$$

with high probability for representation satisfying $\|\pi\| < \sqrt{2}$.

3. Applications to actions on the circle

Navas studied rigidity properties of diffeomorphic actions on the circle.

Vanishing of cohomology for L_p for $p > 2$ improves the differentiability class in his result.

Corollary

Let q be a power of a prime number and G_q be be the corresponding $\widetilde{A}_2$ group. Then every homomorphism $h : G \to \mathrm{Diff}_+^{1+\alpha}(S^1)$ has finite image for

$$\alpha > \frac{\frac{1}{2}\ln(2(q^2+q+1)(q+1)) - \ln(2) - \ln\left(\sqrt{1-\frac{\sqrt{q}}{q+1}}\right)}{\ln(q^2+q+1)+\ln(q+1)}.$$

4. Eigenvalues of the p-Laplacian

The p-Laplacian $\Delta_p : \ell_p(V) \to \ell_p(V)$ is defined by the formula

$$\Delta_p f(x) = \sum_{x \sim y} (f(x) - f(y))^{[p]} \omega(x, y),$$

for $f : V \to \mathbb{R}$, where $a^{[p]} = |a|^{p-1} \operatorname{sign}(a)$.

As mentioned before, in the case when X is isomorphic to the Hilbert space, one of the Poincaré consants can be dropped in the assumptions of the Main Theorem.

In particular, when X is finite dimensional we can use this and the Main Theorem to find a lower bound on the eigenvalue of the p-Laplacian on the Cayley graph of a finite quotient, similarly as using the Kazhdan constant to bound the expanding constant of a finite quotient from below.

Comments

- For uniformly bounded representations, Juhani Koivisto (Ph.D. student at University of Helsinki) extended the theorem of Ballmann and Świątkowski:

Theorem (J. Koivisto, 2012)

Let G be acting properly discontinuously and cocompactly on a 2-dimensional contractible simplicial complex K and denote by $\kappa(x)$ the Poincaré constant of the link of a vertex $x \in K$. If

$$\|\pi\| \leq \frac{\sqrt[4]{2}}{\kappa(x)}$$

for every vertex $x \in K$ then $H^1(G, \pi) = 0$ (i.e., every affine action with linear part π has a fixed point).

- For random group a fixed point theorem for isometric actions on Hadamard spaces was proved by H. Izeki, T. Kondo and S. Nayatani (2009)

Comments

- The range of p for the examples discussed could be improved just by getting better estimates of Poincaré constants.

- Another imporant question in this context is the relation between

$$(\text{g.o. cocycles}_{\pi})^* \quad \text{and} \quad \text{g.o. cocycles}_{\overline{\pi}}$$

When are they isomorphic?
When is the natural map between them surjective/injective?

Here this was guaranteed by the two Poincaré constants, but in general?

Knowing this relation would allow to further extend these techniques.

Comments

Recently, with Kate Juschenko we have found a characterization of Yu's property A in terms of uniformly bounded representations:

Theorem (Juschenko – N.)

A group G has Yu's property A iff there exists a family of uniformly bounded representations on a Hilbert space that converge to the trivial representation.

Recall that property (T) is equivalent to the trivial representation being isolated in the Fell topology among the (equivalence classes of) unitary representations.

A strengthening of property (T) to uniformly bounded representations could allow to find new examples of groups without property A (i.e., non-exact groups).

References

Poincaré inequalities and rigidity for actions on Banach spaces
arXiv:1107.1896

Group actions on Banach spaces (survey)
arXiv:1302.6609