Reductive groups and their representations
Part I: From reductive algebraic groups to finite groups of Lie type

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The Plan

Part I: From reductive algebraic groups to finite groups of Lie type

Part II: Representations in defining characteristic

Part III: Complex representations and generic character tables

Part IV: Small degree representations and some applications

Notation (1)

K: algebraically closed field (here usually $K = \bar{\mathbb{F}}_p$)

Definition. An algebraic group is an algebraic variety with a group structure, s.t. multiplication and inversion are morphisms of varieties.

Examples. $\mathrm{GL}_n(K)$, K^+, $K^\times$, $\mathrm{SL}_n(K)$, $\mathrm{Sp}_{2l}(K)$, upper triangular matrices in $\mathrm{GL}_n(K)$.

Definition. A linear algebraic group over K is an algebraic group that is isomorphic to a closed subgroup of $\mathrm{GL}_n(K)$ for some n.

G: a linear algebraic group over K

Definition. $g \in G$ is called semisimple (unipotent) if $\phi(g)$ is diagonalizable (has all eigenvalues 1) for any $\phi : G \hookrightarrow \mathrm{GL}_n(K)$. (is independent of ϕ)

Notation (2)

G is called

connected: if it is connected as variety in the Zariski topology (the connected component G^0 of G containing 1 is a subgroup of finite index, its cosets are all connected components)

reductive: if its unipotent radical (the largest closed connected unipotent normal subgroup) is trivial

semisimple: if its radical (the largest closed connected solvable normal subgroup) is trivial

simple: if it has no non-trivial closed connected normal subgroup

Examples. from our examples all linear groups are connected, $\mathrm{GL}_n(K)$ is reductive but not semisimple, $\mathrm{SL}_n(K)$ and $\mathrm{Sp}_{2l}(K)$ are simple.

Some groups related to $G = \mathrm{GL}_n(K)$ (a BN-pair)

T: diagonal matrices in G (a maximal torus $\cong (K^\times)^n$)

B: upper triangular matrices in G (a Borel subgroup, solvable)

U_{ij}, $1 \leq i, j \leq n$, $i \neq j$: subgroup $\{u_{ij}(a) = 1 + aE_{ij} \mid a \in K\} \cong K^+$ (a root subgroup, for $t = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in T$ we have $u_{ij}(a)^t = u_{ij}(\lambda_j \lambda_i^{-1} a)$)

$U := \langle U_{ij} \mid i < j \rangle \lhd B$ (unipotent radical of B)

N: the normalizer $N_G(T)$ = subgroup of monomial matrices

$W = N/T$: this is $\cong S_n$, the symmetric group (the Weyl group of G)

We have

- $G = \langle T, U_{i,j} \mid i, j \rangle$,
- $B = T \ltimes U$,
- $T = B \cap N$,
- $\langle U_{ij}, U_{ji} \rangle \cong \mathrm{SL}_2(K)$ for $i \neq j$,
- $G = \dot{\bigcup}_{w \in W} B\dot{w}B$

Tori and (co-)characters

T: a torus, i.e. a linear algebraic group isomorphic to $(K^\times)^r$ for some r

$X = X(T)$: $\mathrm{Hom}(T, K^\times) = \{x : T \to K^\times, (\lambda_1, \ldots, \lambda_r) \mapsto \lambda_1^{n_1} \cdots \lambda_r^{n_r} \mid n_i \in \mathbb{Z}\}$, the character group of T.

$Y = Y(T)$: $\mathrm{Hom}(K^\times, T) = \{y : K^\times \to T, \lambda \mapsto (\lambda^{n_1}, \ldots, \lambda^{n_r}) \mid n_i \in \mathbb{Z}\}$, the cocharacter group of T.

We have

- $X \cong \mathbb{Z}^r$, $Y \cong \mathbb{Z}^r$
- X and Y are dual via $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$, $\langle x, y \rangle = k$ if $x \circ y : K^\times \to K^\times$, $\lambda \mapsto \lambda^k$
- $T \cong Y \otimes_{\mathbb{Z}} K^\times \cong \mathrm{Hom}(X, K^\times)$ as abelian groups

Classification

- Conjugation action of T on root subgroups defines finite subset $\Phi \subset X$, the roots of G.
- Together with a finite subset of coroots $\Phi^\vee \subset Y$ we associate a root datum $(X, \Phi, Y, \Phi^\vee)$ to G (fulfills certain combinatorial conditions).

Isomorphism Theorem. [Chevalley, Steinberg] The root datum and the field K determine G up to isomorphism. They determine a presentation of G, abstract generators labeled by $Y \otimes_{\mathbb{Z}} K^\times$ and $u_\alpha(a)$ for $\alpha \in \Phi$, $a \in K^+$, relations computable from root datum (Steinberg presentation).

Existence Theorem. [Chevalley, Steinberg] Each root datum arises from a connected reductive group over K as above.

Classification (2)

The simple G are classified by the root data corresponding to "irreducible" root systems:

Classical groups:

$\boldsymbol{A_l}$: $\mathrm{SL}_{l+1}(K)$,. . . , $\mathrm{PGL}_{l+1}(K)$ ($l \geq 1$)

$\boldsymbol{B_l}$: $\mathrm{Spin}_{2l+1}(K)$, $\mathrm{SO}_{2l+1}(K)$ ($l \geq 2$)

$\boldsymbol{C_l}$: $\mathrm{Sp}_{2l}(K)$, $\mathrm{PCSp}_{2l}(K)$ ($l \geq 3$)

$\boldsymbol{D_l}$: Spin$2l(K)$, $\mathrm{SO}_{2l}(K)$, $\mathrm{HSpin}_{2l}(K)$ (l even), $\mathrm{PCO}^0_{2l}(K)$ ($l \geq 4$)

Exceptional groups:

$\boldsymbol{G_2(K)}$, $\boldsymbol{F_4(K)}$, $\boldsymbol{E_l(K)}$ (l=6,7,8): two types in cases $E_6(K)$ and $E_7(K)$

Frobenius morphisms

From now p is a prime and $K = \bar{\mathbb{F}}_p$, G is a linear algebraic group over K.

For any power q of p define $F_q : \mathrm{GL}_n(K) \to \mathrm{GL}_n(K)$, $(a_{ij}) \mapsto (a_{ij}^q)$.

Definition. A morphism $F : G \to G$ is a standard Frobenius morphism if there is a q and $\phi : G \hookrightarrow \mathrm{GL}_n(K)$ such that $\phi(F(g)) = F_q(\phi(g))$ for all $g \in G$. And F is a Frobenius morphism if there is a power F^e which is standard with respect to F_{q^e}. If $q = p^f$ is an integer we say that G is defined over $\mathbb{F}_q$ via F.

The group of fixed points $G^F = \{g \in G \mid F(g) = g\}$ is finite.

Definition. If G is a connected reductive group with Frobenius morphism F, then G^F is called a finite group of Lie type.

Examples. $G = \mathrm{SL}_n(\bar{\mathbb{F}}_q)$, $F = F_q$, then $G^F = \mathrm{SL}_n(q)$.
$G = \mathrm{SL}_n(\bar{\mathbb{F}}_q)$, $F(A) := F_q(A^{-tr})$, then $G^F = \mathrm{SU}_n(q)$. $G = F_4(\bar{\mathbb{F}}_2)$, $m \in \mathbb{N}$, there is F with $F^2 = F_{2^{2m+1}}$ and $G^F = {}^2F_4(2^{2m+1})$ are the large Ree groups.

The Lang-Steinberg theorem

From now let G be connected.

Theorem. [Lang-Steinberg] If $F : G \to G$ is a Frobenius morphism then $L : G \to G$, $g \mapsto gF(g^{-1})$, is surjective.

Corollary. Let G act transitively on a set M, and $\tilde{F} : M \to M$ with $\tilde{F}(m)F(g) = \tilde{F}(mg)$ for all $m \in M$, $g \in G$.

- There exists $m_0 \in M$ with $\tilde{F}(m_0) = m_0$. Write $M^{\tilde{F}}$ for set of $\tilde{F}$-stable elements.
  [Prf.: $m \in M$, then $\tilde{F}(m) = mg'$ for some $g' \in G$. Let $g' = gF(g^{-1})$, then $\tilde{F}(mg) = mg'F(g) = mg$.]
- Assume that $H = \mathrm{Stab}_G(m_0)$ is closed, and let $A = A(m_0) = H/H^0$, F induces an automorphism on A. Elements $a, a' \in A$ are F-conjugate if there is $b \in A$ such that $a' = b^{-1}aF(b)$. There is a bijection

$$\begin{array}{rcl} G^F\text{-orbits of } M^{\tilde{F}} & \longrightarrow & F\text{-conjugacy classes of } A \\ G^F\text{-orbit of } m_0 g \in M^{\tilde{F}} & \mapsto & F\text{-conjugacy class of } gF(g^{-1})H^0 \end{array}$$

Applications of Lang-Steinberg

Let G be connected reductive and $F : G \to G$ be a Frobenius morphism.

- All maximal tori T of G are conjugate and F maps maximal tori to maximal tori. Then there is an F-stable maximal torus $T_1 = F(T_1) \leq G$. For $N = N_G(T_1)$ we have $N/N^0 = N/T_1 = W$, the Weyl group of G, and F induces a map $F : W \to W$. We have a bijection
$$G^F\text{-conjugacy classes of } \{T \leq G \mid F(T) = T\} \xrightarrow{\sim} F-\text{conjugacy classes of } W$$
- All Borel subgroups in G are conjugate. F maps Borel subgroups to Borel subgroups. Hence there is an F-stable Borel subgroup $B = F(B) \leq G$. Since $N_G(B) = B$ is connected, all F-stable Borel subgroups are conjugate under G^F.
- Let $g \in G$ and g^G its conjugacy class. Then $F(g^G) = g^G$ if and only if g is conjugate to an element $g_0 \in G^F$. Let $C = C_G(g_0)$. Then the G^F-conjugacy classes in $g^G \cap G^F$ are in bijection with the F-conjugacy classes of C/C^0.
[In $G = \mathrm{GL}_n(\bar{\mathbb{F}}_q)$ always $C = C^0$. Hence, for each $g \in G$ the intersection $g^G \cap G^F$ is either empty or a single G^F-conjugacy class.]

BN-pairs

For G, F as before let $B = F(B)$ a Borel subgroup, $T = F(T) \leq B$ a maximal torus and $N = N_G(T)$. Then G^F is also a group with BN-pair with respect to B^F and N^F.

Jordan decomposition

For each $g \in G$ there is a unique semisimple element $s \in G$ and a unique unipotent element $u \in G$ such that $g = su = ug$.

If $g = su = us \in G^F$ then $s, u \in G^F$; here s is of p'-order and the order of u is a power of p.