Reductive groups and their representations
Part III: Complex representations and generic character tables

Frank Lübeck
Lehrstuhl D für Mathematik, RWTH Aachen

Kraków, April 27-28, 2019

Notation

G: connected reductive group $/K = \bar{\mathbb{F}}_p$

$F : G \to G$ Frobenius morphism with $F^k = F_{q^k}$

G^F finite group of Lie type

We want to know the character table of G^F (irreducible representations over $\mathbb{C}$)

Remark. Algebraic group G has no finite dimensional representation over $\mathbb{C}$.

Dual group

G, F determined by root datum $(X, \Phi, Y, \Phi^\vee)$, K, F

Can exchange roles of X and Y:

Dual group G^*, F is determined by $(Y, \Phi^\vee, X, \Phi)$, K, F

Examples: $G^F = \mathrm{SL}_{l+1}(q)$ and $(G^*)^F = \mathrm{PGL}_{l+1}(q)$,
or $G^F = \mathrm{Spin}_{2l+1}(q)$ (type B_l) and $(G^*)^F = \mathrm{PCSp}_{2l}(q)$ (type C_l)

From Lang-Steinberg we had:

$W/F\text{-conj.} \xrightarrow{\sim} \{F\text{-stable maximal tori } T \leq G\}/G^F, \quad w \mapsto T_w \mod G^F$

$W/F\text{-conj.} \xrightarrow{\sim} \{F\text{-stable maximal tori } T \leq G^*\}/(G^*)^F, w \mapsto T_w^* \mod (G^*)^F$

Can identify $(T_w^*)^F$ with $\mathrm{Irr}(T_w^F)$

Deligne-Lusztig characters

$T = F(T) < B = F(B)$: F-stable maximal torus and Borel subgroup, $U \leq B$

$w \in W = N_G(T)/T$, $\dot{w} \in N$ with $\dot{w}T = w$

$Y(\dot{w}) := \{gU \in G/U \mid g^{-1}F(g) \in U\dot{w}U\}$

G^F operates on $Y(\dot{w})$ from the left

$T^{wF} := \{t \in T \mid {}^{w}F(t) = t\} \tilde{=} T_w^F$ operates on $Y(\dot{w})$ from the right

$\mathcal{H}_w := \sum_{i \geq 0} (-1)^i H_c^i(Y(\dot{w}))$, l-adic cohomology with compact support ($l \neq p$)

This becomes a virtual G^F-T^{wF}-bimodule

Definition. [Deligne-Lusztig] We have a map $R_{T_w}^G$ from virtual $\bar{\mathbb{Q}}_l T^{wF}$-modules to virtual $\bar{\mathbb{Q}}_l G^F$-modules, sending V to $\mathcal{H}_w \otimes_{\bar{\mathbb{Q}}_l T^{wF}} V$.

For $\theta \in \mathrm{Irr}(T^{wF})$ let $s \in (T^*)^{wF}$ be the corresponding element in the dual torus; write $R_{T_w}^G(s)$ for the character of the Deligne-Lusztig induced module of θ.

Properties of Deligne-Lusztig characters $R^G_{T_w}(s)$

- The definition does not depend on choices
- For each $\chi \in \mathrm{Irr}(G^F)$ there exist T, s with $\langle \chi, R^G_{T_w}(s)\rangle \neq 0$
- (Orthogonality) Let $s \in (T^*)^{Fw}$ and $s' \in (T^*)^{Fw'}$, then
  $$\langle R^G_{T_w}(s), R^G_{T_{w'}}(s')\rangle = |\{x \in W \mid xwF(x^{-1}) = w' \text{ and } s' = s^x\}|$$
  In particular $\pm R^G_{T_w}(s)$ is irreducible if the stabilizer of s in W is trivial
- If $R^G_{T_w}(s)$ and $R^G_{T_{w'}}(s')$ have a common irreducible constituent, then s and s' are conjugate in $(G^*)^F$
  This induces a partition of $\mathrm{Irr}(G^F)$, parts labeled by semisimple conjugacy classes of $(G^*)^F$, and called Lusztig series:

$$\mathrm{Irr}(G^F) = \dot{\bigcup}_{s/(G^*)^F \text{ semisimple}} \mathcal{E}(G^F, s)$$

Jordan decomposition of characters

Characters in Lusztig series $\mathcal{E}(G^F, 1)$ for $s = 1$ are called unipotent

Fact: The centralizer of a semisimple element in G is again a reductive group (but in general not connected)

Theorem. [Lusztig, '80s]

(a) Gave combinatorial parameterization of unipotent characters in all cases (E.g., by partitions of n in case $GL_n(q)$), and their multiplicities in the Deligne-Lusztig characters

(b) Showed that parameterization of $\mathcal{E}(G^F, s)$ and multipicities in Deligne-Lusztig characters is the same as for unipotent characters in $C_{G^*}(s)$.

Note that these descriptions are independend of q, only depend on the type of root system of G

Remarks on character values

- Values of Deligne-Lusztig characters are ”computable” if known on unipotent elements (Green functions)
- Often the Deligne-Lusztig characters do not generate the space of class functions
- [Lusztig ’80s] developed theory of character sheaves; addresses both problems: makes Green function computable in many cases and produces a basis of space of class functions
- Parameteration of conjugacy classes in G^F: first semisimple conjugacy classes in G^F and their centralizers, then unipotent classes in each centralizer.
- Parameteration of $\mathrm{Irr}(G^F)$: first semisimple conjugacy classes in $(G^*)^F$ and their centralizers, then unipotent characters in each centralizer.