Reductive groups and their representations
Part IV: Small degree representations and some applications

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Representations of small degree in defining characteristic

Theorem. [Wong, '60s] Fix a type of G (root datum) and a dominant weight λ of G. There is a bilinear form on each weight space of $(V(\lambda)_{\mathbb{Z}})_\mu$, Gram matrix (entries in $\mathbb{Z}$) can be computed by calculations in universal enveloping algebra. Then for each prime p we have $\dim L(\lambda)_\mu =$ rank of Gram matrix modulo p.

Task Given a type of G by a root datum and some bound $m \in \mathbb{N}$. For all characteristics p find all irreducible representations of G in defining characteristic p of degree at most m.

[L. 2001] computed this for certain bounds for each type of rank at most 9, results were repeatedly extended since then

Example:

Type E_8, bound $m = 100000$

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All highest weights λ and primes p such that $L(\lambda)$ in characteristic p has degree at most m.

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000)	all	23125	(00000002)	7	30132	(00000010)	3
248	(00000001)	all	26504	(00000010)	2	30132	(00000010)	5
3626	(10000000)	2	26999	(00000002)	31	30380	(00000010)	$\neq$2,3,5
3875	(10000000)	$\neq$2	27000	(00000002)	$\neq$7,31			

My webpage also contains the characters of all of these representations, e.g.

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Highest weight: [ 0, 0, 0, 0, 0, 0, 1, 0 ]
Characteristic: 3
Dimension: 30132
Level    dominant weight              weight multiplicity  length of W-orbit
115     [ 0, 0, 0, 0, 0, 0, 1, 0 ]       1                 6720
93      [ 1, 0, 0, 0, 0, 0, 0, 0 ]       7                 2160
59      [ 0, 0, 0, 0, 0, 0, 0, 1 ]      34                  240
1       [ 0, 0, 0, 0, 0, 0, 0, 0 ]     132                    1
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Classical types of large rank l are done with $m = l^3$

Example of a character table in case $K = \mathbb{C}$

$G = A_5 = \langle (1,2,3,4,5), (3,4,5) \rangle$, alternating group on 5 points Conjugacy classes: $()^G$, $(1,2)(3,4)^G$, $(1,2,3)^G$, $(1,2,3,4,5)^G$, $(1,2,3,5,4)^G$

Character table (from GAP):

	1a	2a	3a	5a	5b
X.1	1	1	1	1	1
X.2	3	-1	.	A	*A
X.3	3	-1	.	*A	A
X.4	4	.	1	-1	-1
X.5	5	1	-1	.	.

where $\mathtt{A} = (1+\sqrt{5})/2$ and $*\mathtt{A} = (1-\sqrt{5})/2$

Generic character table of $SL_2(q)$ with $q = 2^k$

$SL_2(q)$	C_1	C_2	$C_3(a)$	$C_4(a)$
χ_1	1	1	1	1
χ_2	q	0	1	-1
$\chi_3(n)$	$q+1$	1	$\zeta_1^{an} + \zeta_1^{-an}$	0
$\chi_4(n)$	$q-1$	-1	0	$-\xi_1^{an} - \xi_1^{-an}$

$\zeta_1 := \exp(\frac{2\pi\sqrt{-1}}{q-1})$, $\quad \xi_1 := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

Parameter ranges:

$\chi_3(n)$:	$n = 1, \ldots, q-2$	($\frac{1}{2}(q-2)$ characters)
$\chi_4(n)$:	$n = 1, \ldots, q$	($\frac{1}{2}q$ characters)
$C_3(a)$:	$a = 1, \ldots, q-2$	($\frac{1}{2}(q-2)$ classes)
$C_4(a)$:	$a = 1, \ldots, q$	($\frac{1}{2}q$ classes)

(BTW: $A_5 \cong SL_2(4)$)

Generic table for $SL_2(q)$ with odd q

$SL_2(q)$	$C_1(i)$	$C_2(i)$
χ_1	1	1
χ_2	q	0
χ_3	$\frac{1}{2}(q+1)(-1)^{\frac{1}{2}(q-1)i}$	$\frac{1}{2}(-1)^{\frac{1}{2}(q-1)i} + \frac{1}{2}\sqrt{q}\varepsilon_4^{\frac{1}{2}(q-1)}$
χ_4	$\frac{1}{2}(q+1)(-1)^{\frac{1}{2}(q-1)i}$	$\frac{1}{2}(-1)^{\frac{1}{2}(q-1)i} - \frac{1}{2}\sqrt{q}\varepsilon_4^{\frac{1}{2}(q-1)}$
χ_5	$\frac{1}{2}(q-1)(-1)^{\frac{1}{2}qi+\frac{1}{2}i}$	$-\frac{1}{2}(-1)^{\frac{1}{2}(q+1)i} + \frac{1}{2}\sqrt{q}\varepsilon_4^{2i+\frac{1}{2}q-\frac{1}{2}}$
χ_6	$\frac{1}{2}(q-1)(-1)^{\frac{1}{2}qi+\frac{1}{2}i}$	$-\frac{1}{2}(-1)^{\frac{1}{2}(q+1)i} - \frac{1}{2}\sqrt{q}\varepsilon_4^{2i+\frac{1}{2}q-\frac{1}{2}}$
$\chi_7(k)$	$(q+1)(-1)^{ik}$	$(-1)^{ik}$
$\chi_8(k)$	$(q-1)(-1)^{ik}$	$-(-1)^{ik}$

$\varepsilon_4 := \exp(\frac{2\pi\sqrt{-1}}{4}), \quad \zeta_1 := \exp(\frac{2\pi\sqrt{-1}}{q-1}), \quad \xi_1 := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

Generic table for $SL_2(q)$ with odd q (cont.)

$SL_2(q)$	$C_3(i)$	$C_4(i)$	$C_5(i)$
χ_1	1	1	1
χ_2	0	1	-1
χ_3	$\frac{1}{2}(-1)^{\frac{1}{2}(q-1)i} - \frac{1}{2}\sqrt{q}\varepsilon_4^{\frac{1}{2}(q-1)}$	$(-1)^i$	0
χ_4	$\frac{1}{2}(-1)^{\frac{1}{2}(q-1)i} + \frac{1}{2}\sqrt{q}\varepsilon_4^{\frac{1}{2}(q-1)}$	$(-1)^i$	0
χ_5	$-\frac{1}{2}(-1)^{\frac{1}{2}(q+1)i} - \frac{1}{2}\sqrt{q}\varepsilon_4^{2i+\frac{1}{2}q-\frac{1}{2}}$	0	$-(-1)^i$
χ_6	$-\frac{1}{2}(-1)^{\frac{1}{2}(q+1)i} + \frac{1}{2}\sqrt{q}\varepsilon_4^{2i+\frac{1}{2}q-\frac{1}{2}}$	0	$-(-1)^i$
$\chi_7(k)$	$(-1)^{ik}$	$\zeta_1^{ik} + \zeta_1^{-ik}$	0
$\chi_8(k)$	$-(-1)^{ik}$	0	$-\xi_1^{ik} - \xi_1^{-ik}$

$\varepsilon_4 := \exp(\frac{2\pi\sqrt{-1}}{4}), \quad \zeta_1 := \exp(\frac{2\pi\sqrt{-1}}{q-1}), \quad \xi_1 := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

Observations about generic character tables

Fix type of G by root datum

- There are finitely many types of centralizers of semisimple elements
- The number of unipotent classes is finite
- The smallest non-trivial degree is a ”polynomial in q”
- Many numbers here are PORC (polynomial on residue classes)
- For $E_8(q)$ the generic character tables have > 10000 columns and rows, largest entries have $|W|$ summands with 8 parameters for classes and characters each
- Some applications only need parts of the table,
- e.g., my webpage has lists of all (generic) irreducible character degree for simple groups up to rank 9