Irreducible euclidean representations of the Fibonacci groups

Rafał Lutowski

Institute of Mathematics, University of Gdańsk

16th Andrzej Jankowski Memorial Lecture
Mini Conference

Euclidean and affine maps in $\mathbb{R}^n$

- $E(n) = \mathrm{Iso}(\mathbb{R}^n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$ – the group of isometries of the Euclidean space $\mathbb{R}^n$.
- $A(n) = \mathrm{Aff}(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ – the group of affine maps of $\mathbb{R}^n$.

Remark

1. $E(n) \subset A(n)$.
2. $A(n) = \{(A, a) \mid A \in \mathrm{GL}(n, \mathbb{R}), a \in \mathbb{R}^n\}$ and
$$\forall_{(A,a),(B,b)\in A(n)} (A, a)(B, b) = (AB, Ab + a).$$
3. The action of the group $A(n)$ ($E(n)$) on $\mathbb{R}^n$:
$$\forall_{(A,a)\in A(n)} \forall_{x\in\mathbb{R}^n} (A, a) \cdot x = Ax + a.$$

Euclidean and affine maps in $\mathbb{R}^n$

Remark

1. $E(n) \subset A(n)$.
2. $A(n) = \{(A, a) \mid A \in \mathrm{GL}(n, \mathbb{R}), a \in \mathbb{R}^n\}$ and
$$\forall_{(A,a),(B,b)\in A(n)} (A, a)(B, b) = (AB, Ab + a).$$
3. The action of the group $A(n)$ ($E(n)$) on $\mathbb{R}^n$:
$$\forall_{(A,a)\in A(n)} \forall_{x\in\mathbb{R}^n} (A, a) \cdot x = Ax + a.$$

Lemma

There is a faithful representation $A(n) \to \mathrm{GL}_{n+1}(\mathbb{R})$ *given by*

$$\forall_{(A,a)\in A(n)} \ (A, a) \mapsto \begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix}.$$

Crystallographic and Bieberbach groups

Definition

A group Γ is an n-dimensional crystallographic group if it is a discrete and cocompact subgroup of $E(n)$. If in addition Γ is torsionfree then we call it a Bieberbach group.

Remark

If $\Gamma \subset E(n)$ is a Bieberbach group then $X = \mathbb{R}^n/\Gamma$ is a flat manifold and $\pi_1(X) \cong \Gamma$.

Crystallographic and Bieberbach groups

Definition

A group Γ is an n-dimensional crystallographic group if it is a discrete and cocompact subgroup of $E(n)$. If in addition Γ is torsionfree then we call it a Bieberbach group.

Remark

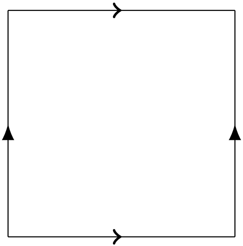
If $\Gamma \subset E(n)$ is a Bieberbach group then $X = \mathbb{R}^n/\Gamma$ is a flat manifold and $\pi_1(X) \cong \Gamma$.

Two dimensional flat manifolds

Torus

$$\Gamma_1 = \langle (I, e_1), (I, e_2) \rangle$$

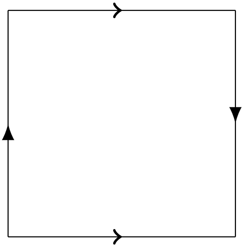
$$\mathbb{R}^2/\Gamma_1 :$$



Klein bottle

$$\Gamma_2 = \left\langle \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \right), (I, e_2) \right\rangle$$

$$\mathbb{R}^2/\Gamma_2 :$$



Bieberbach theorems

Theorem (Bieberbach 1911, 1912)

1. *Let $\Gamma \subset E(n)$ be an n-dimensional crystallographic group. The subgroup $\Gamma \cap (1 \times \mathbb{R}^n)$ of pure translations of Γ is free abelian group of rank n. Moreover it is maximal abelian normal subgroup of Γ of finite index.*
2. *For every $n \in \mathbb{N}$ there are a finite number of isomorphism classes of crystallogrpahic groups of dimension n.*
3. *Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group $A(n)$.*

Bieberbach theorems

Theorem (Bieberbach 1911, 1912)

1. *Let $\Gamma \subset E(n)$ be an n-dimensional crystallographic group. The subgroup $\Gamma \cap (1 \times \mathbb{R}^n)$ of pure translations of Γ is free abelian group of rank n. Moreover it is maximal abelian normal subgroup of Γ of finite index.*
2. *For every $n \in \mathbb{N}$ there are a finite number of isomorphism classes of crystallogrpahic groups of dimension n.*
3. *Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group $A(n)$.*

Bieberbach theorems

Theorem (Bieberbach 1911, 1912)

1. *Let $\Gamma \subset E(n)$ be an n-dimensional crystallographic group. The subgroup $\Gamma \cap (1 \times \mathbb{R}^n)$ of pure translations of Γ is free abelian group of rank n. Moreover it is maximal abelian normal subgroup of Γ of finite index.*
2. *For every $n \in \mathbb{N}$ there are a finite number of isomorphism classes of crystallogrpahic groups of dimension n.*
3. *Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group $A(n)$.*

Structure of Bieberbach groups

Let $\Gamma \subset E(n)$ be a Bieberbach group and $X = \mathbb{R}^n/\Gamma$.

- Γ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- G – finite group – holonomy group of Γ (of X).
- We get a holonomy representation $\varphi\colon G \to \mathrm{GL}_n(\mathbb{Z})$:

$$\forall_{z \in \mathbb{Z}^n \subset \Gamma} \forall_{g \in G}\ \varphi_g(z) = \overline{g} z \overline{g}^{-1},$$

where $\pi(\overline{g}) = g$.
- φ is $\mathbb{R}$-equivalent to the holonomy representation of X.

Structure of Bieberbach groups

Let $\Gamma \subset E(n)$ be a Bieberbach group and $X = \mathbb{R}^n/\Gamma$.

- Γ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- G – finite group – holonomy group of Γ (of X).
- We get a holonomy representation $\varphi\colon G \to \mathrm{GL}_n(\mathbb{Z})$:

$$\forall_{z \in \mathbb{Z}^n \subset \Gamma} \forall_{g \in G}\ \varphi_g(z) = \overline{g} z \overline{g}^{-1},$$

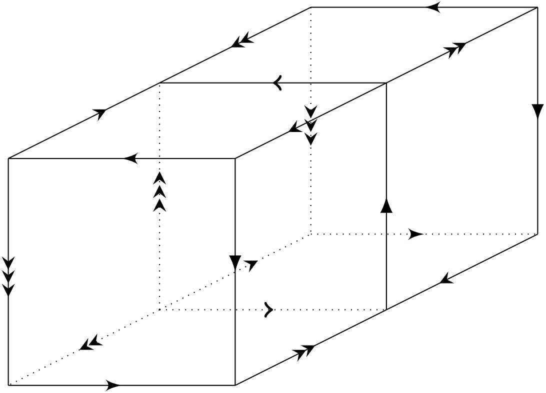
  where $\pi(\overline{g}) = g$.
- φ is $\mathbb{R}$-equivalent to the holonomy representation of X.

Classical Hantzsche-Wendt manifold

$$\Gamma = \left\langle \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right) \right\rangle \subset E(3)$$

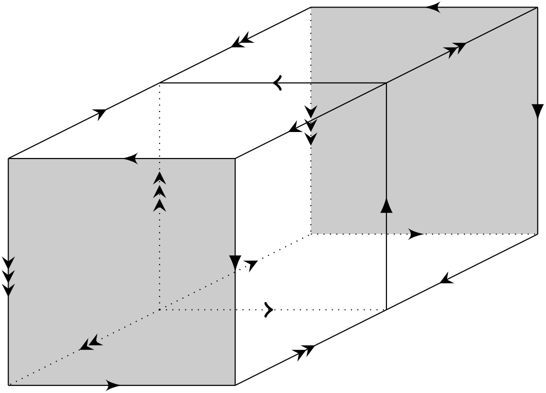
Classical Hantzsche-Wendt manifold

$$\Gamma = \left\langle \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right) \right\rangle \subset E(3)$$



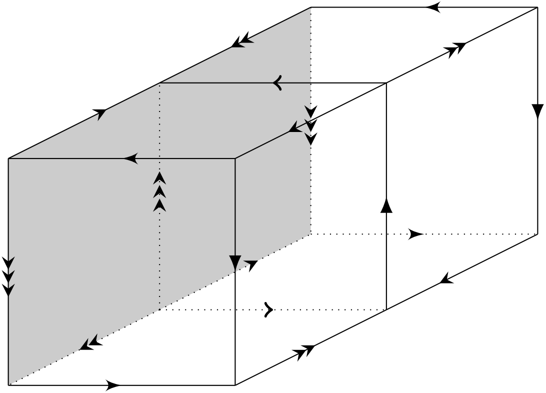
Classical Hantzsche-Wendt manifold

$$\Gamma = \left\langle \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right) \right\rangle \subset E(3)$$



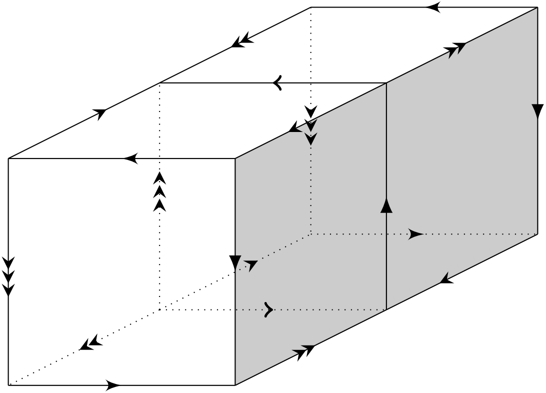
Classical Hantzsche-Wendt manifold

$$\Gamma = \left\langle \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right) \right\rangle \subset E(3)$$



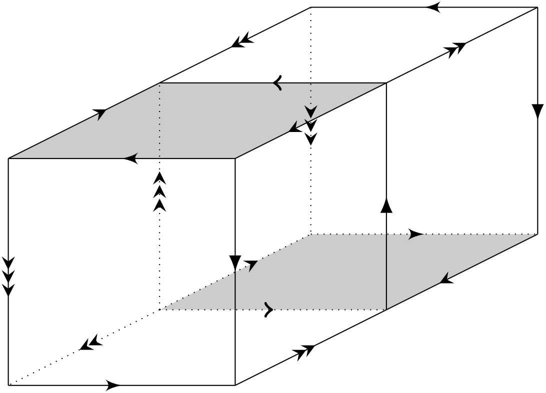
Classical Hantzsche-Wendt manifold

$$\Gamma = \left\langle \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right) \right\rangle \subset E(3)$$



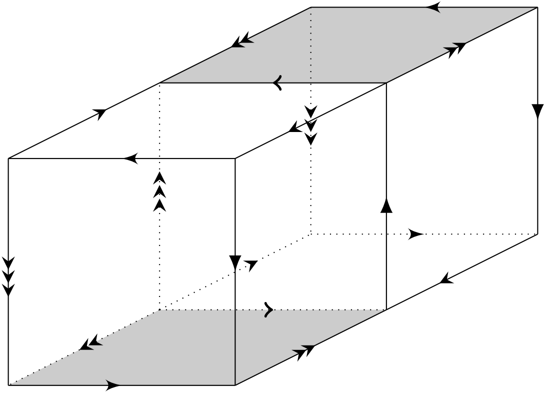
Classical Hantzsche-Wendt manifold

$$\Gamma = \left\langle \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right) \right\rangle \subset E(3)$$



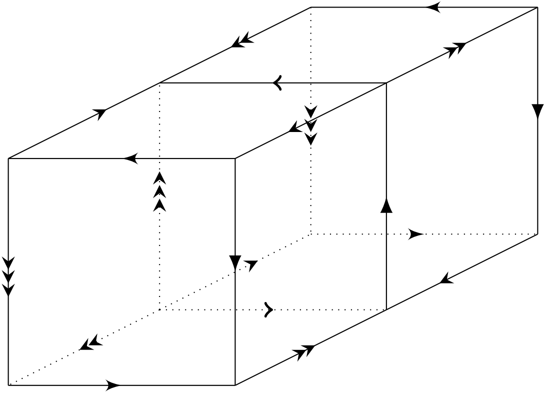
Classical Hantzsche-Wendt manifold

$$\Gamma = \left\langle \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right) \right\rangle \subset E(3)$$



Classical Hantzsche-Wendt manifold

$$\Gamma = \left\langle \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right) \right\rangle \subset E(3)$$



Hantzsche-Wendt manifolds and groups

Definition

Let $n \in \mathbb{N}$ be odd. Let $\Gamma \subset E(n)$ be a Bieberbach group with holonomy group G and holonomy representation $\varphi\colon G \to \mathrm{GL}_n(\mathbb{Z})$. If

$$G \cong (\mathbb{Z}_2)^{n-1} \quad \text{and} \quad \varphi(G) \subset \mathrm{SL}_n(\mathbb{Z})$$

then Γ is called a Hantzsche-Wendt group (HW-group) and $X = \mathbb{R}^n/\Gamma$ – a Hantzsche-Wendt manifold (HW-manifold).

Structure of HW-groups

Theorem (Rosetti, Szczepański 2005)

Let $n \in \mathbb{N}$ *be odd. Let* Γ *be an* n *dimensional HW-group. Then*

$$\Gamma \cong \langle (B_i, b_i) \mid i = 1, \ldots, n \rangle \subset E(n),$$

where

$$B_i = \operatorname{diag}(\underbrace{-1, \ldots, -1}_{i-1}, 1, -1, \ldots, -1)$$

and $b_i \in \frac{1}{2}\mathbb{Z}^n$ *for* $i = 1, \ldots, n$*.*

Remark

The group $\langle (B_i, b_i) \mid i = 1, \ldots, n \rangle$ is a Bieberbach group, hence it is a HW-group.

Structure of HW-groups

Theorem (Rosetti, Szczepański 2005)

Let $n \in \mathbb{N}$ *be odd. Let* Γ *be an* n *dimensional HW-group. Then*

$$\Gamma \cong \langle (B_i, b_i) \mid i = 1, \ldots, n \rangle \subset E(n),$$

where

$$B_i = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{i-1}, 1, -1, \ldots, -1)$$

and $b_i \in \frac{1}{2}\mathbb{Z}^n$ *for* $i = 1, \ldots, n$.

Remark

The group $\langle (B_i, b_i) \mid i = 1, \ldots, n \rangle$ is a Bieberbach group, hence it is a HW-group.

Encoding HW-groups

- $\Gamma \subset E(n)$ – HW-group.
- $\Gamma \cong \langle (B_i, b_i) \mid i = 1, \ldots, n \rangle$.

Corollary

Up to isomorphism every HW-group is defined by a matrix

$$\begin{bmatrix} b_1 & \ldots & b_n \end{bmatrix} \in M_n(\tfrac{1}{2}\mathbb{Z}).$$

Fibonacci groups

Definition

Let $r, n \in \mathbb{N}$. The Fibonacci group $F(r,n)$ is a group with presentation

$$\begin{aligned} F(r,n) = \langle a_0, a_1, \ldots, a_{n-1} \mid & \, a_0 a_1 \ldots a_{r-1} = a_r, \\ & \, a_1 a_2 \ldots a_r = a_{r+1}, \\ & \, \vdots \\ & \, a_{n-1} a_0 \ldots a_{r-2} = a_{r-1} \rangle. \end{aligned}$$

Global assumption

The subscripts will always be taken modulo n.

Fibonacci groups

Definition

Let $r, n \in \mathbb{N}$. The Fibonacci group $F(r, n)$ is a group with presentation

$$F(r,n) = \langle a_0, a_1, \dots, a_{n-1} \mid a_0 a_1 \dots a_{r-1} = a_r, \\ a_1 a_2 \dots a_r = a_{r+1}, \\ \vdots \\ a_{n-1} a_0 \dots a_{r-2} = a_{r-1} \rangle.$$

Global assumption

The subscripts will always be taken modulo n.

Fibonacci groups in geometry

Proposition

Let X be the 3 dimensional HW-manifold. Then

$$\pi_1(X) \cong F(2,6).$$

Theorem (Helling, Kim, Mennicke 1998)

For $n \geq 4$ there exists a closed hyperbolic manifold X such that

$$\pi_1(X) \cong F(2,2n).$$

Theorem (Szczepański, Vesnin 2000)

For odd $n \in \mathbb{N}$ the Fibonacci group $F(2,n)$ cannot be a fundamental group of any hyperbolic manifold of finite volume.

Fibonacci groups in geometry

Proposition

Let X be the 3 dimensional HW-manifold. Then

$$\pi_1(X) \cong F(2,6).$$

Theorem (Helling, Kim, Mennicke 1998)

For $n \geq 4$ there exists a closed hyperbolic manifold X such that

$$\pi_1(X) \cong F(2,2n).$$

Theorem (Szczepański, Vesnin 2000)

For odd $n \in \mathbb{N}$ the Fibonacci group $F(2,n)$ cannot be a fundamental group of any hyperbolic manifold of finite volume.

Fibonacci groups in geometry

Proposition

Let X be the 3 dimensional HW-manifold. Then

$$\pi_1(X) \cong F(2,6).$$

Theorem (Helling, Kim, Mennicke 1998)

For $n \geq 4$ there exists a closed hyperbolic manifold X such that

$$\pi_1(X) \cong F(2,2n).$$

Theorem (Szczepański, Vesnin 2000)

For odd $n \in \mathbb{N}$ the Fibonacci group $F(2,n)$ cannot be a fundamental group of any hyperbolic manifold of finite volume.

Fibonacci and Hantzsche-Wendt groups

Theorem (Szczepański 2001)

Let $n \in \mathbb{N}$ be odd, $n \geq 3$. Let $\Gamma \subset E(n)$ be a HW-group defined by the matrix

$$\begin{bmatrix} \frac{1}{2} & 0 & \dots & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \ddots & \vdots & \vdots & 0 \\ 0 & \frac{1}{2} & \ddots & 0 & 0 & \vdots \\ 0 & 0 & \ddots & \frac{1}{2} & 0 & 0 \\ \vdots & \vdots & \ddots & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then there exists an epimorphism

$$\Phi\colon F(n-1, 2n) \to \Gamma.$$

Cyclic HW-groups

Let $\Gamma \subset E(n)$ be as in the previous theorem.

- Γ is "cyclic" because of the form of the matrix which defines it.
- Γ is "cyclic" because it has generators which behave exactly as the generators of some Fibonacci group.

Question

Is every HW-group cyclic in the second sense?

More precisely:

Question

Is every HW-group an epimorphic image of some Fibonacci group?

Cyclic HW-groups

Let $\Gamma \subset E(n)$ be as in the previous theorem.

- Γ is "cyclic" because of the form of the matrix which defines it.
- Γ is "cyclic" because it has generators which behave exactly as the generators of some Fibonacci group.

Question

Is every HW-group cyclic in the second sense?

More precisely:

Question

Is every HW-group an epimorphic image of some Fibonacci group?

Cyclic HW-groups

Let $\Gamma \subset E(n)$ be as in the previous theorem.

- Γ is "cyclic" because of the form of the matrix which defines it.
- Γ is "cyclic" because it has generators which behave exactly as the generators of some Fibonacci group.

Question

Is every HW-group cyclic in the second sense?

More precisely:

Question

Is every HW-group an epimorphic image of some Fibonacci group?

Cyclic HW-groups

Let $\Gamma \subset E(n)$ be as in the previous theorem.

- Γ is "cyclic" because of the form of the matrix which defines it.
- Γ is "cyclic" because it has generators which behave exactly as the generators of some Fibonacci group.

Question

Is every HW-group cyclic in the second sense?

More precisely:

Question

Is every HW-group an epimorphic image of some Fibonacci group?

Cyclic HW-groups

Let $\Gamma \subset E(n)$ be as in the previous theorem.

- Γ is "cyclic" because of the form of the matrix which defines it.
- Γ is "cyclic" because it has generators which behave exactly as the generators of some Fibonacci group.

Question

Is every HW-group cyclic in the second sense?

More precisely:

Question

Is every HW-group an epimorphic image of some Fibonacci group?

Euclidean representation

Definition

An euclidean representation of a group G is any homomorphism $\varphi\colon G \to E(n)$ for some $n \in \mathbb{N}$.

Example

Let Γ be an n-dimensional cyclic HW-group. Then

$$\Phi\colon F(n-1, 2n) \to \Gamma \subset E(n)$$

is an euclidean representation of the Fibonacci group $F(n-1, 2n)$.

Euclidean representation

Definition

An euclidean representation of a group G is any homomorphism $\varphi\colon G \to E(n)$ for some $n \in \mathbb{N}$.

Example

Let Γ be an n-dimensional cyclic HW-group. Then

$$\Phi\colon F(n-1, 2n) \to \Gamma \subset E(n)$$

is an euclidean representation of the Fibonacci group $F(n-1, 2n)$.

Decomposability and irreducibility

Example

Let $G = \langle a, b \mid [a,b] = 1 \rangle$ be a free abelian group of rank 2. Let $\varphi_1, \varphi_2 \colon G \to E(1)$ be euclidean representations of the group G defined by

$$\varphi_1(a) = (1,1), \quad \varphi_1(b) = (1,0),$$
$$\varphi_2(a) = (1,0), \quad \varphi_1(b) = (1,1).$$

We get an euclidean representation $\varphi_1 \oplus \varphi_2 \colon G \to E(2)$:

$$\varphi_1 \oplus \varphi_2(a) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \quad \varphi_1 \oplus \varphi_2(b) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

The above action of G on $V = \mathbb{R} \oplus \mathbb{R}$ is defined as a direct sum, but V does not have proper invariant subspace under this action!

Decomposability and irreducibility

Example

Let $G = \langle a, b \mid [a, b] = 1\rangle$ be a free abelian group of rank 2. Let $\varphi_1, \varphi_2 \colon G \to E(1)$ be euclidean representations of the group G defined by

$$\varphi_1(a) = (1, 1), \quad \varphi_1(b) = (1, 0),$$
$$\varphi_2(a) = (1, 0), \quad \varphi_1(b) = (1, 1).$$

We get an euclidean representation $\varphi_1 \oplus \varphi_2 \colon G \to E(2)$:

$$\varphi_1 \oplus \varphi_2(a) = \left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right), \quad \varphi_1 \oplus \varphi_2(b) = \left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right).$$

The above action of G on $V = \mathbb{R} \oplus \mathbb{R}$ is defined as a direct sum, but V does not have proper invariant subspace under this action!

Decomposability and irreducibility

Example

Let $G = \langle a, b \mid [a, b] = 1 \rangle$ be a free abelian group of rank 2. Let $\varphi_1, \varphi_2 \colon G \to E(1)$ be euclidean representations of the group G defined by

$$\varphi_1(a) = (1, 1), \quad \varphi_1(b) = (1, 0),$$
$$\varphi_2(a) = (1, 0), \quad \varphi_1(b) = (1, 1).$$

We get an euclidean representation $\varphi_1 \oplus \varphi_2 \colon G \to E(2)$:

$$\varphi_1 \oplus \varphi_2(a) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \quad \varphi_1 \oplus \varphi_2(b) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

The above action of G on $V = \mathbb{R} \oplus \mathbb{R}$ is defined as a direct sum, but V does not have proper invariant subspace under this action!

Decomposition of euclidean representations

- $n \in \mathbb{N}$.
- $\pi\colon E(n) \to \mathrm{O}(n)$ – given by $\pi(B, b) = B$, $(B, b) \in E(n)$.
- $\varphi\colon G \to E(n)$ – euclidean representation of a group G.
- $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k$ – decomposition of $\pi\varphi\colon G \to \mathrm{O}(n)$.
- $p_i\colon \mathbb{R}^n \to V_i$ – projections, $i = 1, \ldots, n$.

Proposition

We have

$$\varphi = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(k)},$$

where for every $1 \leq i \leq n$, $\varphi^{(i)}\colon G \to \mathrm{Iso}(V_i)$ *is given by*

$$\forall_{v \in V_i}\ \varphi^{(i)}_g(v) = (A, p_i(a))v = Av + p_i(a),$$

where $g \in G$ *and* $(A, a) = \varphi_g$.

Decomposition of euclidean representations

- $n \in \mathbb{N}$.
- $\pi\colon E(n) \to \mathrm{O}(n)$ – given by $\pi(B,b) = B$, $(B,b) \in E(n)$.
- $\varphi\colon G \to E(n)$ – euclidean representation of a group G.
- $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k$ – decomposition of $\pi\varphi\colon G \to \mathrm{O}(n)$.
- $p_i\colon \mathbb{R}^n \to V_i$ – projections, $i = 1, \ldots, n$.

Proposition

We have

$$\varphi = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(k)},$$

where for every $1 \leq i \leq n$, $\varphi^{(i)}\colon G \to \mathrm{Iso}(V_i)$ *is given by*

$$\forall_{v \in V_i}\ \varphi^{(i)}_g(v) = (A, p_i(a))v = Av + p_i(a),$$

where $g \in G$ *and* $(A,a) = \varphi_g$.

Decomposition of euclidean representations

- $n \in \mathbb{N}$.
- $\pi \colon E(n) \to \mathrm{O}(n)$ – given by $\pi(B, b) = B$, $(B, b) \in E(n)$.
- $\varphi \colon G \to E(n)$ – euclidean representation of a group G.
- $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k$ – decomposition of $\pi\varphi \colon G \to \mathrm{O}(n)$.
- $p_i \colon \mathbb{R}^n \to V_i$ – projections, $i = 1, \ldots, n$.
- $\varphi = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(k)}$.

Remark

If $G = \Gamma \subset E(n)$ is a crystallographic group, $\varphi = id_\Gamma$, then $\pi\varphi(\Gamma)$ is a finite group, hence the decomposition

$$\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k$$

can be made in such a way that V_i is irreducible, for $i = 1, \ldots, k$.

Decomposition of HW-groups

Corollary

Let $n \in \mathbb{N}$ be odd. Let $\Gamma \subset E(n)$ be an n-dimensional HW-group defined by a matrix $[b_{ij}]_{1 \leq i,j \leq n}$. Then

$$id_\Gamma = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(n)},$$

where homomorphisms $\varphi^{(i)} \colon \Gamma \to E(1)$ are given by

$$\forall_{1 \leq j \leq n} \; \varphi^{(i)}(B_j, b_j) = \left((-1)^{\delta_{ij}+1}, b_{ij} \right).$$

Decomposition of HW-groups

$$id_\Gamma = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(n)}$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \right)$$

$$\shortparallel$$

$$(-1, b_{1,j}) \oplus \ldots \oplus (-1, b_{j-1,j}) \oplus (1, b_{j,j}) \oplus (-1, b_{j+1,j}) \oplus \ldots \oplus (-1, b_{n,j})$$

Decomposition of HW-groups

$$id_\Gamma = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(n)}$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \right)$$

$$\shortparallel$$

$$(-1, b_{1,j}) \oplus \ldots \oplus (-1, b_{j-1,j}) \oplus (1, b_{j,j}) \oplus (-1, b_{j+1,j}) \oplus \ldots \oplus (-1, b_{n,j})$$

Decomposition of HW-groups

$$id_\Gamma = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(n)}$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \right)$$

$$\shortparallel$$

$$(-1, b_{1,j}) \oplus \ldots \oplus (-1, b_{j-1,j}) \oplus (1, b_{j,j}) \oplus (-1, b_{j+1,j}) \oplus \ldots \oplus (-1, b_{n,j})$$

Decomposition of HW-groups

$$id_\Gamma = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(n)}$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \right)$$

$$\shortparallel$$

$$(-1, b_{1,j}) \oplus \ldots \oplus (-1, b_{j-1,j}) \oplus (1, b_{j,j}) \oplus (-1, b_{j+1,j}) \oplus \ldots \oplus (-1, b_{n,j})$$

Decomposition of HW-groups

$$id_\Gamma = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(n)}$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \right)$$

$$\shortparallel$$

$$(-1, b_{1,j}) \oplus \ldots \oplus (-1, b_{j-1,j}) \oplus (1, b_{j,j}) \oplus (-1, b_{j+1,j}) \oplus \ldots \oplus (-1, b_{n,j})$$

Decomposition of HW-groups

$$id_\Gamma = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(n)}$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \right)$$

$$\shortparallel$$

$$(-1, b_{1,j}) \oplus \ldots \oplus (-1, b_{j-1,j}) \oplus (1, b_{j,j}) \oplus (-1, b_{j+1,j}) \oplus \ldots \oplus (-1, b_{n,j})$$

Decomposition of HW-groups

$$id_\Gamma = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(n)}$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \right)$$

$$\shortparallel$$

$$(-1, b_{1,j}) \oplus \ldots \oplus (-1, b_{j-1,j}) \oplus (1, b_{j,j}) \oplus (-1, b_{j+1,j}) \oplus \ldots \oplus (-1, b_{n,j})$$

Shift automorphism

Lemma

Let $r, n \in \mathbb{N}$. Let $F(r,n)$ be the Fibonacci group. Then the homomorphism $\sigma\colon F(r,n) \to F(r,n)$ defined by

$$\forall_{0 \leq i \leq n-1}\ \sigma(a_i) = a_{i-1}$$

is an automorphism of $F(r,n)$.

Remark

Let's call σ the left shift automorphism of $F(r,n)$.

One dimensional euclidean representations

Theorem

Let $n \in \mathbb{N}$ *be odd. Let* $\Gamma = \langle C_i \mid i = 0, \ldots, n-2 \rangle \subset E(1)$*, where*

$$C_0 = (1, c_0), C_1 = (-1, c_1), \ldots, C_{n-2} = (-1, c_{n-2})$$

and $c_i \in \mathbb{R}$ *for* $i = 0, \ldots, n-2$*. Then there exists an epimorphism*

$$\varphi \colon F(n-1, 2n) \to \Gamma$$

such that

$$\varphi(a_i) = C_i$$

for $i = 0, \ldots, n-2$ *and* $a_0, \ldots, a_{2n}$ *are the "cyclic" generators of* $F(n-1, 2n)$*.*

Proof

We will show that the sequence (C_i) of elements of Γ, defined recursively by

$$\forall_{i \geq n-1} C_i = C_{i-n+1} C_{i-n+2} \ldots C_{i-1}$$

is periodic with period $2n$. For this to prove it is enough to show that

$$C_{2n} = C_0, C_{2n+1} = C_1, \ldots, C_{3n-2} = C_{n-2}.$$

Note that for $i > n-1$ we have

$$\begin{aligned} C_i &= C_{i-n+1} C_{i-n+2} \ldots C_{i-1} \\ &= C_{i-n}^{-1} (C_{i-n} C_{i-n+1} C_{i-n+2} \ldots C_{i-2}) C_{i-1} = C_{i-n} C_{i-1}^2. \end{aligned}$$

Proof

We will show that the sequence (C_i) of elements of Γ, defined recursively by

$$\forall_{i \geq n-1} C_i = C_{i-n+1} C_{i-n+2} \dots C_{i-1}$$

is periodic with period $2n$. For this to prove it is enough to show that

$$C_{2n} = C_0, C_{2n+1} = C_1, \dots, C_{3n-2} = C_{n-2}.$$

Note that for $i > n-1$ we have

$$\begin{aligned} C_i &= C_{i-n+1} C_{i-n+2} \dots C_{i-1} \\ &= C_{i-n}^{-1} (C_{i-n} C_{i-n+1} C_{i-n+2} \dots C_{i-2}) C_{i-1} = C_{i-n} C_{i-1}^2. \end{aligned}$$

Proof

$$\forall_{i>n-1}\; C_i = C_{i-n} C_{i-1}^2.$$

$$\begin{aligned}
C_{n-1} &= (1, c_0)(-1, c_1)\dots(-1, c_{n-2}) && = (-1, c_{n-1}) \\
C_n &= C_0^{-1} C_{n-1}^2 = (1, c_0)^{-1}(-1, c_{n-1})^2 && = (1, -c_0) \\
C_{n+1} &= C_1^{-1} C_n^2 = (-1, c_1)(1, -2c_0) && = (-1, 2c_0 + c_1) \\
C_{n+i} &= C_i^{-1} C_{n+i-1}^2 && = C_i,\; 2 \le i \le n-1 \\
C_{2n} &= C_n^{-1} C_{2n-1}^2 = (1, -c_0)^{-1} && = C_0 \\
C_{2n+1} &= C_{n+1}^{-1} C_{2n}^2 = (-1, 2c_0 + c_1)(1, 2c_0) && = C_1 \\
C_{2n+i} &= C_{n+i}^{-1} C_{2n+i-1}^2 = C_{n+i} && = C_i,\; 2 \le i \le n-1
\end{aligned}$$

$$\forall_{0\le i\le n-2}\; C_{2n+i} = C_i.$$

Proof

$$\forall_{i>n-1}\, C_i = C_{i-n}C_{i-1}^2.$$

$$
\begin{aligned}
C_{n-1} &= (1, c_0)(-1, c_1)\dots(-1, c_{n-2}) && = (-1, c_{n-1}) \\
C_n &= C_0^{-1}C_{n-1}^2 = (1, c_0)^{-1}(-1, c_{n-1})^2 && = (1, -c_0) \\
C_{n+1} &= C_1^{-1}C_n^2 = (-1, c_1)(1, -2c_0) && = (-1, 2c_0 + c_1) \\
C_{n+i} &= C_i^{-1}C_{n+i-1}^2 && = C_i,\ 2 \le i \le n-1 \\
C_{2n} &= C_n^{-1}C_{2n-1}^2 = (1, -c_0)^{-1} && = C_0 \\
C_{2n+1} &= C_{n+1}^{-1}C_{2n}^2 = (-1, 2c_0 + c_1)(1, 2c_0) && = C_1 \\
C_{2n+i} &= C_{n+i}^{-1}C_{2n+i-1}^2 = C_{n+i} && = C_i,\ 2 \le i \le n-1
\end{aligned}
$$

$$\forall_{0\le i\le n-2}\, C_{2n+i} = C_i.$$

Proof

$$\forall_{i>n-1}\ C_i = C_{i-n}C_{i-1}^2.$$

$$\begin{aligned}
C_{n-1} &= (1,c_0)(-1,c_1)\dots(-1,c_{n-2}) &&= (-1,c_{n-1})\\
C_n &= C_0^{-1}C_{n-1}^2 = (1,c_0)^{-1}(-1,c_{n-1})^2 &&= (1,-c_0)\\
C_{n+1} &= C_1^{-1}C_n^2 = (-1,c_1)(1,-2c_0) &&= (-1,2c_0+c_1)\\
C_{n+i} &= C_i^{-1}C_{n+i-1}^2 &&= C_i,\ 2\le i\le n-1\\
C_{2n} &= C_n^{-1}C_{2n-1}^2 = (1,-c_0)^{-1} &&= C_0\\
C_{2n+1} &= C_{n+1}^{-1}C_{2n}^2 = (-1,2c_0+c_1)(1,2c_0) &&= C_1\\
C_{2n+i} &= C_{n+i}^{-1}C_{2n+i-1}^2 = C_{n+i} &&= C_i,\ 2\le i\le n-1
\end{aligned}$$

$$\forall_{0\le i\le n-2}\ C_{2n+i} = C_i.$$

Proof

$$\forall_{i>n-1}\ C_i = C_{i-n}C_{i-1}^2.$$

$$\begin{aligned}
C_{n-1} &= (1, c_0)(-1, c_1)\dots(-1, c_{n-2}) && = (-1, c_{n-1}) \\
C_n &= C_0^{-1}C_{n-1}^2 = (1, c_0)^{-1}(-1, c_{n-1})^2 && = (1, -c_0) \\
C_{n+1} &= C_1^{-1}C_n^2 = (-1, c_1)(1, -2c_0) && = (-1, 2c_0 + c_1) \\
C_{n+i} &= C_i^{-1}C_{n+i-1}^2 && = C_i,\ 2 \le i \le n-1 \\
C_{2n} &= C_n^{-1}C_{2n-1}^2 = (1, -c_0)^{-1} && = C_0 \\
C_{2n+1} &= C_{n+1}^{-1}C_{2n}^2 = (-1, 2c_0 + c_1)(1, 2c_0) && = C_1 \\
C_{2n+i} &= C_{n+i}^{-1}C_{2n+i-1}^2 = C_{n+i} && = C_i,\ 2 \le i \le n-1
\end{aligned}$$

$$\forall_{0\le i\le n-2}\ C_{2n+i} = C_i.$$

Proof

$$\forall_{i>n-1}\ C_i = C_{i-n}C_{i-1}^2.$$

$$\begin{aligned}
C_{n-1} &= (1,c_0)(-1,c_1)\ldots(-1,c_{n-2}) && = (-1,c_{n-1})\\
C_n &= C_0^{-1}C_{n-1}^2 = (1,c_0)^{-1}(-1,c_{n-1})^2 && = (1,-c_0)\\
C_{n+1} &= C_1^{-1}C_n^2 = (-1,c_1)(1,-2c_0) && = (-1,2c_0+c_1)\\
C_{n+i} &= C_i^{-1}C_{n+i-1}^2 && = C_i,\ 2 \le i \le n-1\\
C_{2n} &= C_n^{-1}C_{2n-1}^2 = (1,-c_0)^{-1} && = C_0\\
C_{2n+1} &= C_{n+1}^{-1}C_{2n}^2 = (-1,2c_0+c_1)(1,2c_0) && = C_1\\
C_{2n+i} &= C_{n+i}^{-1}C_{2n+i-1}^2 = C_{n+i} && = C_i,\ 2 \le i \le n-1
\end{aligned}$$

$$\forall_{0\le i\le n-2}\ C_{2n+i} = C_i.$$

Proof

$$\forall_{i>n-1}\ C_i = C_{i-n}C_{i-1}^2.$$

$$\begin{aligned}
C_{n-1} &= (1,c_0)(-1,c_1)\dots(-1,c_{n-2}) && = (-1,c_{n-1})\\
C_n &= C_0^{-1}C_{n-1}^2 = (1,c_0)^{-1}(-1,c_{n-1})^2 && = (1,-c_0)\\
C_{n+1} &= C_1^{-1}C_n^2 = (-1,c_1)(1,-2c_0) && = (-1,2c_0+c_1)\\
C_{n+i} &= C_i^{-1}C_{n+i-1}^2 && = C_i,\ 2\le i\le n-1\\
C_{2n} &= C_n^{-1}C_{2n-1}^2 = (1,-c_0)^{-1} && = C_0\\
C_{2n+1} &= C_{n+1}^{-1}C_{2n}^2 = (-1,2c_0+c_1)(1,2c_0) && = C_1\\
C_{2n+i} &= C_{n+i}^{-1}C_{2n+i-1}^2 = C_{n+i} && = C_i,\ 2\le i\le n-1
\end{aligned}$$

$$\forall_{0\le i\le n-2}\ C_{2n+i} = C_i.$$

Proof

$$\forall_{i>n-1}\ C_i = C_{i-n}C_{i-1}^2.$$

$$\begin{aligned}
C_{n-1} &= (1,c_0)(-1,c_1)\dots(-1,c_{n-2}) && = (-1,c_{n-1})\\
C_n &= C_0^{-1}C_{n-1}^2 = (1,c_0)^{-1}(-1,c_{n-1})^2 && = (1,-c_0)\\
C_{n+1} &= C_1^{-1}C_n^2 = (-1,c_1)(1,-2c_0) && = (-1,2c_0+c_1)\\
C_{n+i} &= C_i^{-1}C_{n+i-1}^2 && = C_i,\ 2\le i\le n-1\\
C_{2n} &= C_n^{-1}C_{2n-1}^2 = (1,-c_0)^{-1} && = C_0\\
C_{2n+1} &= C_{n+1}^{-1}C_{2n}^2 = (-1,2c_0+c_1)(1,2c_0) && = C_1\\
C_{2n+i} &= C_{n+i}^{-1}C_{2n+i-1}^2 = C_{n+i} && = C_i,\ 2\le i\le n-1
\end{aligned}$$

$$\forall_{0\le i\le n-2}\ C_{2n+i} = C_i.$$

Proof

$$\forall_{i>n-1}\ C_i = C_{i-n} C_{i-1}^2.$$

$$\begin{aligned}
C_{n-1} &= (1, c_0)(-1, c_1)\dots(-1, c_{n-2}) && = (-1, c_{n-1}) \\
C_n &= C_0^{-1} C_{n-1}^2 = (1, c_0)^{-1}(-1, c_{n-1})^2 && = (1, -c_0) \\
C_{n+1} &= C_1^{-1} C_n^2 = (-1, c_1)(1, -2c_0) && = (-1, 2c_0 + c_1) \\
C_{n+i} &= C_i^{-1} C_{n+i-1}^2 && = C_i,\ 2 \le i \le n-1 \\
C_{2n} &= C_n^{-1} C_{2n-1}^2 = (1, -c_0)^{-1} && = C_0 \\
C_{2n+1} &= C_{n+1}^{-1} C_{2n}^2 = (-1, 2c_0 + c_1)(1, 2c_0) && = C_1 \\
C_{2n+i} &= C_{n+i}^{-1} C_{2n+i-1}^2 = C_{n+i} && = C_i,\ 2 \le i \le n-1
\end{aligned}$$

$$\forall_{0\le i\le n-2}\ C_{2n+i} = C_i.$$

Proof

$$\forall_{i>n-1}\ C_i = C_{i-n}C_{i-1}^2.$$

$$\begin{array}{lcll}
C_{n-1} & = & (1, c_0)(-1, c_1)\dots(-1, c_{n-2}) & = (-1, c_{n-1}) \\
C_n & = & C_0^{-1}C_{n-1}^2 = (1, c_0)^{-1}(-1, c_{n-1})^2 & = (1, -c_0) \\
C_{n+1} & = & C_1^{-1}C_n^2 = (-1, c_1)(1, -2c_0) & = (-1, 2c_0 + c_1) \\
C_{n+i} & = & C_i^{-1}C_{n+i-1}^2 & = C_i,\ 2 \le i \le n-1 \\
C_{2n} & = & C_n^{-1}C_{2n-1}^2 = (1, -c_0)^{-1} & = C_0 \\
C_{2n+1} & = & C_{n+1}^{-1}C_{2n}^2 = (-1, 2c_0 + c_1)(1, 2c_0) & = C_1 \\
C_{2n+i} & = & C_{n+i}^{-1}C_{2n+i-1}^2 = C_{n+i} & = C_i,\ 2 \le i \le n-1
\end{array}$$

$$\forall_{0\le i\le n-2}\ C_{2n+i} = C_i.$$

Every HW-group is cyclic

Theorem

Let $n \in \mathbb{N}$ be odd. Let $\Gamma \subset E(n)$ be a HW-group. Then there exists an epimorphism

$$\Phi\colon F(n-1, 2n) \to \Gamma.$$

Proof

Decomposition of HW-group

- Let
$$[b_{ij}]_{0\leq i,j<n}$$
be a matrix of Γ.
- Let
$$id_\Gamma = \varphi^{(0)} \oplus \ldots \oplus \varphi^{(n-1)}$$
be the euclidean decomposition of id_Γ.
- For every $0 \leq i < n$ there exists epimorphism
$$f_i \colon F(n-1, 2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$$
given by
$$f_i(a_0) = (1, b_{ii}), f_i(a_1) = (-1, b_{i,i+1}), \ldots, f_i(a_{n-1}) = (-1, b_{i,i+n-1}).$$

Proof

Decomposition of HW-group

- Let
$$\left[b_{ij}\right]_{0\leq i,j<n}$$
be a matrix of Γ.
- Let
$$id_\Gamma = \varphi^{(0)} \oplus \ldots \oplus \varphi^{(n-1)}$$
be the euclidean decomposition of id_Γ.
- For every $0 \leq i < n$ there exists epimorphism
$$f_i \colon F(n-1, 2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$$
given by
$$f_i(a_0) = (1, b_{ii}), f_i(a_1) = (-1, b_{i,i+1}), \ldots, f_i(a_{n-1}) = (-1, b_{i,i+n-1}).$$

Proof

Decomposition of HW-group

- Let

$$\left[b_{ij}\right]_{0\leq i,j<n}$$

  be a matrix of Γ.
- Let

$$id_\Gamma = \varphi^{(0)} \oplus \ldots \oplus \varphi^{(n-1)}$$

  be the euclidean decomposition of id_Γ.
- For every $0 \leq i < n$ there exists epimorphism

$$f_i \colon F(n-1, 2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$$

  given by

$$f_i(a_0) = (1, b_{ii}), f_i(a_1) = (-1, b_{i,i+1}), \ldots, f_i(a_{n-1}) = (-1, b_{i,i+n-1}).$$

Proof

Decomposition of HW-group

- Let

$$\left[b_{ij}\right]_{0\leq i,j<n}$$

  be a matrix of Γ.
- Let

$$id_\Gamma = \varphi^{(0)} \oplus \ldots \oplus \varphi^{(n-1)}$$

  be the euclidean decomposition of id_Γ.
- For every $0 \leq i < n$ there exists epimorphism

$$f_i \colon F(n-1, 2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$$

  given by

$$\forall_{0\leq j<n}\ f_i(a_j) = \left((-1)^{1+\delta_{i,i+j}}, b_{i,i+j}\right)$$

Proof

Left shift automorphism

- $\sigma \in \mathrm{Aut}(F(n-1,2n))$ – left shift automorphism.
- $f_i\sigma^i\colon F(n-1,2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$ for every $0 \le i < n$.
- $$\forall_{0\le i,j<n}\ f_i\sigma^i(a_j) = f_i(a_{j-i}) = \left((-1)^{1+\delta_{i,j}}, b_{i,j}\right)$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix}\right)$$

Proof

Left shift automorphism

- $\sigma \in \mathrm{Aut}(F(n-1, 2n))$ – left shift automorphism.
- $f_i\sigma^i \colon F(n-1, 2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$ for every $0 \leq i < n$.
- $$\forall_{0 \leq i,j < n}\ f_i\sigma^i(a_j) = f_i(a_{j-i}) = \left((-1)^{1+\delta_{i,j}}, b_{i,j}\right)$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \right)$$

Proof

Left shift automorphism

- $\sigma \in \mathrm{Aut}(F(n-1, 2n))$ – left shift automorphism.
- $f_i\sigma^i \colon F(n-1, 2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$ for every $0 \leq i < n$.
- $$\forall_{0 \leq i,j < n}\ f_i\sigma^i(a_j) = f_i(a_{j-i}) = \left((-1)^{1+\delta_{i,j}}, b_{i,j}\right)$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix}\right)$$

Proof

Left shift automorphism

- $\sigma \in \mathrm{Aut}(F(n-1,2n))$ – left shift automorphism.
- $f_i\sigma^i \colon F(n-1,2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$ for every $0 \leq i < n$.
- $$\forall_{0 \leq i,j < n}\ f_i\sigma^i(a_j) = f_i(a_{j-i}) = \left((-1)^{1+\delta_{i,j}}, b_{i,j}\right)$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \right)$$

Proof

Left shift automorphism

- $\sigma \in \mathrm{Aut}(F(n-1,2n))$ – left shift automorphism.
- $f_i\sigma^i \colon F(n-1,2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$ for every $0 \leq i < n$.
- $$\forall_{0\leq i,j<n}\ f_i\sigma^i(a_j) = f_i(a_{j-i}) = \left((-1)^{1+\delta_{i,j}}, b_{i,j}\right)$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix}\right) \quad \begin{matrix} f_0\sigma^0(a_j) \\ \vdots \\ f_{j-1}\sigma^{j-1}(a_j) \\ f_j\sigma^j(a_j) \\ f_{j+1}\sigma^{j+1}(a_j) \\ \vdots \\ f_n\sigma^n(a_j) \end{matrix}$$

Proof

Left shift automorphism

- $\sigma \in \mathrm{Aut}(F(n-1,2n))$ – left shift automorphism.
- $f_i\sigma^i \colon F(n-1,2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$ for every $0 \le i < n$.
- $$\forall_{0\le i,j<n}\ f_i\sigma^i(a_j) = f_i(a_{j-i}) = \left((-1)^{1+\delta_{i,j}}, b_{i,j}\right)$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix}\right) \quad \begin{matrix} f_0\sigma^0(a_j) \\ \vdots \\ f_{j-1}\sigma^{j-1}(a_j) \\ f_j\sigma^j(a_j) \\ f_{j+1}\sigma^{j+1}(a_j) \\ \vdots \\ f_n\sigma^n(a_j) \end{matrix}$$

Proof

Left shift automorphism

- $\sigma \in \mathrm{Aut}(F(n-1,2n))$ – left shift automorphism.
- $f_i\sigma^i \colon F(n-1,2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$ for every $0 \leq i < n$.
- $$\forall_{0 \leq i,j < n}\ f_i\sigma^i(a_j) = f_i(a_{j-i}) = \left((-1)^{1+\delta_{i,j}}, b_{i,j}\right)$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix}\right) \quad \begin{matrix} f_0\sigma^0(a_j) \\ \vdots \\ f_{j-1}\sigma^{j-1}(a_j) \\ f_j\sigma^j(a_j) \\ f_{j+1}\sigma^{j+1}(a_j) \\ \vdots \\ f_n\sigma^n(a_j) \end{matrix}$$

Proof

The epimorphism

The map

$$\Phi = \bigoplus_{i=0}^{n-1} f_i \sigma^i$$

is the desired epimorphism:

$$\begin{aligned}\forall_{0\leq j<n}\ \Phi(a_j) =& \bigoplus_{i=0}^{n-1} f_i \sigma^i(a_j) \\ =&(-1, b_{0,j}) \oplus \ldots (-1, b_{j-1,j}) \oplus (1, b_{j,j}) \oplus \\ &\oplus (-1, b_{j+1,j}) \oplus \ldots \oplus (-1, b_{n,j}) \\ =&(B_j, b_j).\end{aligned}$$

Thank you!