Spin structures on flat manifolds

Rafał Lutowski

Institute of Mathematics, University of Gdańsk

Andrzej Jankowski Memorial Lecture
Mini Conference
May 15 - 17, 2015

joint work with Bartosz Putrycz

Bieberbach groups and flat manifolds

- $E(n) = O(n) \ltimes \mathbb{R}^n$ – isometry group of the Euclidean space $\mathbb{R}^n$.
- Discrete and cocompact subgroup $\Gamma \subset E(n)$ – crystallographic group.
- Torsionfree crystallographic $\Gamma \subset E(n)$ – Bieberbach group.
  - $X = \mathbb{R}^n/\Gamma$ – flat manifold (closed connected Riemannian n-manifold with zero sectional curvature).
  - $\pi_1(X) \cong \Gamma$.

Bieberbach groups and flat manifolds

- $E(n) = O(n) \ltimes \mathbb{R}^n$ – isometry group of the Euclidean space $\mathbb{R}^n$.
- Discrete and cocompact subgroup $\Gamma \subset E(n)$ – crystallographic group.
- Torsionfree crystallographic $\Gamma \subset E(n)$ – Bieberbach group.
  - $X = \mathbb{R}^n / \Gamma$ – flat manifold (closed connected Riemannian n-manifold with zero sectional curvature).
  - $\pi_1(X) \cong \Gamma$.

Bieberbach theorems

Theorem (Bieberbach 1911, 1912)

Let $\Gamma \subset E(n)$ *be a crystallographic group.*

1. *The subgroup* $\Gamma \cap (1 \times \mathbb{R}^n)$ *of pure translations of* Γ *is free abelian group of rank* n. *Moreover it is maximal abelian normal subgroup of* Γ *of finite index.*
2. *A crystallographic group is isomorphic to* Γ *if and only if it is conjugate to* Γ *in the group* $A(n) = \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$.
3. *There are a finite number of isomorphic classes of crystallographic groups in each dimension.*

Structure of crystallographic groups

1st Bieberbach theorem

- Γ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- $\pi\colon \Gamma \to \mathrm{SO}(n)$:

$$\forall_{(A,a)\in\Gamma}\ \pi(A,a) = A.$$

- $G = \pi(G)$ – finite group – holonomy group of Γ (X).
- We get a holonomy representation $\varphi\colon G \to \mathrm{GL}(n,\mathbb{Z})$:

$$\varphi_g(z) = \overline{g} z \overline{g}^{-1},$$

  where $z \in \mathbb{Z}^n \subset \Gamma, g \in G, \pi(\overline{g}) = g$.
- φ is $\mathbb{R}$-equivalent to $id\colon G \to G \subset \mathrm{SO}(n)$.

Structure of crystallographic groups

1st Bieberbach theorem

- Γ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- $\pi \colon \Gamma \to \mathrm{SO}(n)$:

$$\forall_{(A,a)\in\Gamma}\ \pi(A,a) = A.$$

- $G = \pi(G)$ – finite group – holonomy group of Γ (X).
- We get a holonomy representation $\varphi \colon G \to \mathrm{GL}(n,\mathbb{Z})$:

$$\varphi_g(z) = \overline{g} z \overline{g}^{-1},$$

where $z \in \mathbb{Z}^n \subset \Gamma, g \in G, \pi(\overline{g}) = g$.
- φ is $\mathbb{R}$-equivalent to $id \colon G \to G \subset \mathrm{SO}(n)$.

Structure of crystallographic groups

1st Bieberbach theorem

- Γ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- $\pi \colon \Gamma \to \mathrm{SO}(n)$:

$$\forall_{(A,a)\in\Gamma}\ \pi(A,a) = A.$$

- $G = \pi(G)$ – finite group – holonomy group of Γ (X).
- We get a holonomy representation $\varphi \colon G \to \mathrm{GL}(n, \mathbb{Z})$:

$$\varphi_g(z) = \overline{g} z \overline{g}^{-1},$$

where $z \in \mathbb{Z}^n \subset \Gamma, g \in G, \pi(\overline{g}) = g$.

- φ is $\mathbb{R}$-equivalent to $id \colon G \to G \subset \mathrm{SO}(n)$.

Structure of crystallographic groups

1st Bieberbach theorem

- Γ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- $\pi\colon \Gamma \to \mathrm{SO}(n)$:

$$\forall_{(A,a)\in\Gamma}\ \pi(A,a) = A.$$

- $G = \pi(G)$ – finite group – holonomy group of Γ (X).
- We get a holonomy representation $\varphi\colon G \to \mathrm{GL}(n,\mathbb{Z})$:

$$\varphi_g(z) = \overline{g}z\overline{g}^{-1},$$

where $z \in \mathbb{Z}^n \subset \Gamma, g \in G, \pi(\overline{g}) = g$.

- φ is $\mathbb{R}$-equivalent to $id\colon G \to G \subset \mathrm{SO}(n)$.

Structure of crystallographic groups

1st Bieberbach theorem

- Γ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- $\pi \colon \Gamma \to \mathrm{SO}(n)$:

$$\forall_{(A,a)\in\Gamma}\ \pi(A,a) = A.$$

- $G = \pi(G)$ – finite group – holonomy group of Γ (X).
- We get a holonomy representation $\varphi \colon G \to \mathrm{GL}(n,\mathbb{Z})$:

$$\varphi_g(z) = \overline{g}z\overline{g}^{-1},$$

  where $z \in \mathbb{Z}^n \subset \Gamma, g \in G, \pi(\overline{g}) = g$.
- φ is $\mathbb{R}$-equivalent to $id \colon G \to G \subset \mathrm{SO}(n)$.

Enumeration of crystallographic groups

3rd Bieberbach theorem

Dimension 2: 17 groups (proof of completeness in 1891);
2 Bieberbach groups.

Dimension 3: 219 groups (Fedorov and Schönflies 1890's);
10 Bieberbach groups.

Dimension 4: 4 783 groups (Brown et al. 1978);
74 Bieberbach groups.

Dimension 5: 222 018 groups (Plesken, Schultz 2000);
1 060 Bieberbach groups.

Dimension 6: 28 927 915 groups (Plesken, Schultz 2000);
38 746 Bieberbach groups.

Enumeration of crystallographic groups

3rd Bieberbach theorem

Dimension 2: 17 groups (proof of completeness in 1891);
2 Bieberbach groups.

Dimension 3: 219 groups (Fedorov and Schönflies 1890's);
10 Bieberbach groups.

Dimension 4: 4 783 groups (Brown et al. 1978);
74 Bieberbach groups.

Dimension 5: 222 018 groups (Plesken, Schultz 2000);
1 060 Bieberbach groups.

Dimension 6: 28 927 915 groups (Plesken, Schultz 2000);
38 746 Bieberbach groups.

Enumeration of crystallographic groups

3rd Bieberbach theorem

Dimension 2: 17 groups (proof of completeness in 1891);
2 Bieberbach groups.

Dimension 3: 219 groups (Fedorov and Schönflies 1890's);
10 Bieberbach groups.

Dimension 4: 4 783 groups (Brown et al. 1978);
74 Bieberbach groups.

Dimension 5: 222 018 groups (Plesken, Schultz 2000);
1 060 Bieberbach groups.

Dimension 6: 28 927 915 groups (Plesken, Schultz 2000);
38 746 Bieberbach groups.

Enumeration of crystallographic groups

3rd Bieberbach theorem

Dimension 2: 17 groups (proof of completeness in 1891);
2 Bieberbach groups.

Dimension 3: 219 groups (Fedorov and Schönflies 1890's);
10 Bieberbach groups.

Dimension 4: 4 783 groups (Brown et al. 1978);
74 Bieberbach groups.

Dimension 5: 222 018 groups (Plesken, Schultz 2000);
1 060 Bieberbach groups.

Dimension 6: 28 927 915 groups (Plesken, Schultz 2000);
38 746 Bieberbach groups.

Enumeration of crystallographic groups

3rd Bieberbach theorem

Dimension 2: 17 groups (proof of completeness in 1891);
2 Bieberbach groups.

Dimension 3: 219 groups (Fedorov and Schönflies 1890's);
10 Bieberbach groups.

Dimension 4: 4 783 groups (Brown et al. 1978);
74 Bieberbach groups.

Dimension 5: 222 018 groups (Plesken, Schultz 2000);
1 060 Bieberbach groups.

Dimension 6: 28 927 915 groups (Plesken, Schultz 2000);
38 746 Bieberbach groups.

Clifford algebra

Definition

Let $n \in \mathbb{N}$. The Clifford algebra C_n is a real associative algebra with one, generated by elements $e_1, \ldots, e_n$, which satisfy relations:

$$\forall_{1 \leq i < j \leq n} \; e_i^2 = -1 \text{ and } e_i e_j = -e_j e_i.$$

- $C_0 = \mathbb{R}, C_1 = \mathbb{C}, C_2 = \mathbb{H}$.
- $\mathbb{R}^n = \operatorname{span}\{e_1, \ldots, e_n\} \subset C_n$.

Clifford algebra

Definition

Let $n \in \mathbb{N}$. The Clifford algebra C_n is a real associative algebra with one, generated by elements $e_1, \dots, e_n$, which satisfy relations:

$$\forall_{1 \leq i < j \leq n}\ e_i^2 = -1 \text{ and } e_i e_j = -e_j e_i.$$

- $C_0 = \mathbb{R}, C_1 = \mathbb{C}, C_2 = \mathbb{H}$.
- $\mathbb{R}^n = \operatorname{span}\{e_1, \dots, e_n\} \subset C_n$.

Three involutions

1. Defined on the generators of the vector space C_n:

$$(e_{i_1} \dots e_{i_k})^* = e_{i_k} \dots e_{i_1}.$$

2. Defined on the generators of the algebra C_n:

$$e_i' = -e_i.$$

3. The composition of the above two:

$$\forall_{a \in C_n} \; \overline{a} = (a')^*.$$

Three involutions

1. Defined on the generators of the vector space C_n:

$$(e_{i_1} \dots e_{i_k})^* = e_{i_k} \dots e_{i_1}.$$

2. Defined on the generators of the algebra C_n:

$$e_i' = -e_i.$$

3. The composition of the above two:

$$\forall_{a \in C_n} \ \overline{a} = (a')^*.$$

Three involutions

1. Defined on the generators of the vector space C_n:
$$(e_{i_1} \dots e_{i_k})^* = e_{i_k} \dots e_{i_1}.$$
2. Defined on the generators of the algebra C_n:
$$e_i' = -e_i.$$
3. The composition of the above two:
$$\forall_{a \in C_n}\ \overline{a} = (a')^*.$$

Spin group

Definition

Let $n \in \mathbb{N}$.

$$\mathrm{Spin}(n) := \{x \in C_n \mid x' = x \wedge x\bar{x} = 1\}.$$

Spin group

Universal cover of special orthogonal group

Proposition

Let $n \in \mathbb{N}$. *The map* $\lambda_n \colon \operatorname{Spin}(n) \to \operatorname{SO}(n)$, *defined by*

$$\lambda_n(x)v = xv\overline{x},$$

where $x \in \operatorname{Spin}(n), v \in \mathbb{R}^n$ *is a continuous group epimorphism.*

For $n \geq 3$:

- $\operatorname{Spin}(n)$ – universal cover of $\operatorname{SO}(n)$.
- $\pi_1(\operatorname{SO}(n)) \cong \mathbb{Z}/2\mathbb{Z}$.
- $\ker \lambda_n = \{\pm 1\}$. We get a central extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow \operatorname{Spin}(n) \xrightarrow{\lambda_n} \operatorname{SO}(n) \longrightarrow 1.$$

Spin group

Universal cover of special orthogonal group

Proposition

Let $n \in \mathbb{N}$. The map $\lambda_n \colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$, defined by

$$\lambda_n(x)v = xv\overline{x},$$

where $x \in \mathrm{Spin}(n), v \in \mathbb{R}^n$ is a continuous group epimorphism.

For $n \geq 3$:

- $\mathrm{Spin}(n)$ – universal cover of $\mathrm{SO}(n)$.
- $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2\mathbb{Z}$.
- $\ker \lambda_n = \{\pm 1\}$. We get a central extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(n) \xrightarrow{\lambda_n} \mathrm{SO}(n) \longrightarrow 1.$$

Spin structures on manifolds

- X – orientable closed manifold of dimension n.
- Q – principal $\mathrm{SO}(n)$-tangent bundle of X.

Definition

A spin structure on X is a pair (P, Λ):

- P – principal $\mathrm{Spin}(n)$-bundle over X;
- $\Lambda \colon P \to Q$ is a 2-fold covering with the commutative diagram:

$$\begin{array}{ccccc} P \times \mathrm{Spin}(n) & \longrightarrow & P & & \\ \Big\downarrow \Lambda \times \lambda_n & & \Lambda \Big\downarrow & \searrow & \\ & & & & X \\ & & & \nearrow & \\ Q \times \mathrm{SO}(n) & \longrightarrow & Q & & \end{array}$$

(The maps in the rows are defined by the action of $\mathrm{Spin}(n)$ and $\mathrm{SO}(n)$ on P and Q respectively.)

Spin structures on manifolds

- X – orientable closed manifold of dimension n.
- Q – principal $\mathrm{SO}(n)$-tangent bundle of X.

Definition

A spin structure on X is a pair (P, Λ):

- P – principal $\mathrm{Spin}(n)$-bundle over X;
- $\Lambda\colon P \to Q$ is a 2-fold covering with the commutative diagram:

$$
\begin{array}{ccccc}
P \times \mathrm{Spin}(n) & \longrightarrow & P & & \\
\Big\downarrow \Lambda \times \lambda_n & & \Lambda \Big\downarrow & \searrow & \\
 & & & & X \\
 & & & \nearrow & \\
Q \times \mathrm{SO}(n) & \longrightarrow & Q & &
\end{array}
$$

(The maps in the rows are defined by the action of $\mathrm{Spin}(n)$ and $\mathrm{SO}(n)$ on P and Q respectively.)

Spin structures on manifolds

- X – orientable closed manifold of dimension n.
- Q – principal $\mathrm{SO}(n)$-tangent bundle of X.

Definition

A spin structure on X is a pair (P, Λ):

- P – principal $\mathrm{Spin}(n)$-bundle over X;
- $\Lambda\colon P \to Q$ is a 2-fold covering with the commutative diagram:

$$
\begin{array}{ccccc}
P \times \mathrm{Spin}(n) & \longrightarrow & P & & \\
\Big\downarrow \Lambda \times \lambda_n & & \Lambda \Big\downarrow & \searrow & \\
 & & & & X \\
 & & & \nearrow & \\
Q \times \mathrm{SO}(n) & \longrightarrow & Q & &
\end{array}
$$

(The maps in the rows are defined by the action of $\mathrm{Spin}(n)$ and $\mathrm{SO}(n)$ on P and Q respectively.)

Example

Every compact oriented 3 manifold admits a spin structure (it has trivial tangent bundle).

Proposition

An orientable closed manifold X has a spin structure if and only if its second Stiefel-Whitney class vanishes:

$$w_2(X) = 0.$$

Moreover in this case spin structures on M are classified by

$$H^1(X, \mathbb{Z}_2).$$

Example

Every compact oriented 3 manifold admits a spin structure (it has trivial tangent bundle).

Proposition

An orientable closed manifold X has a spin structure if and only if its second Stiefel-Whitney class vanishes:

$$w_2(X) = 0.$$

Moreover in this case spin structures on M are classified by

$$H^1(X, \mathbb{Z}_2).$$

Example

Every compact oriented 3 manifold admits a spin structure (it has trivial tangent bundle).

Proposition

An orientable closed manifold X has a spin structure if and only if its second Stiefel-Whitney class vanishes:

$$w_2(X) = 0.$$

Moreover in this case spin structures on M are classified by

$$H^1(X, \mathbb{Z}_2).$$

Spin structures on flat manifolds

Algebraic condition

- $X = \mathbb{R}^n/\Gamma$ – orientable flat n-manifold:

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- $\pi\colon \Gamma \to \mathrm{SO}(n)$.

Proposition (Pfaffle 1999)

The set of spin structures on X is in bijection with the set of the homomorphisms of the form $\varepsilon\colon \Gamma \to \mathrm{Spin}(n)$ which satisfy $\lambda_n \varepsilon = \pi$:

$$\begin{array}{ccc} & & \mathrm{Spin}(n) \\ & \overset{\varepsilon}{\nearrow} & \downarrow \lambda_n \\ \Gamma & \xrightarrow{\pi} & \mathrm{SO}(n) \end{array}$$

Spin structures on flat manifolds

Algebraic condition

- $X = \mathbb{R}^n/\Gamma$ – orientable flat n-manifold:

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

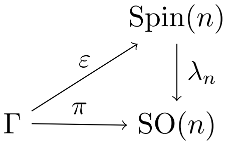
- $\pi\colon \Gamma \to \mathrm{SO}(n)$.

Proposition (Pfaffle 1999)

The set of spin structures on X is in bijection with the set of the homomorphisms of the form $\varepsilon\colon \Gamma \to \mathrm{Spin}(n)$ which satisfy $\lambda_n \varepsilon = \pi$:

$$\begin{array}{ccc} & & \mathrm{Spin}(n) \\ & \overset{\varepsilon}{\nearrow} & \downarrow \lambda_n \\ \Gamma & \xrightarrow{\pi} & \mathrm{SO}(n) \end{array}$$

Determining spin structures



- Every crystallographic group is finitely presented. Let

$$\Gamma = \langle S \mid R \rangle,$$

be a presentation of Γ, with finite S and R.

Determining spin structures

For every map $\varepsilon\colon S \to \mathrm{Spin}(n)$ for which $\lambda_n \varepsilon = \pi$ check if it preserves the relations of Γ:

$$r_1 \dots r_l \in R \overset{?}{\Rightarrow} \varepsilon(r_1) \dots \varepsilon(r_l) = 1,$$

where $r_1, \dots, r_l \in S \cup S^{-1}$.

Determining spin structures

$$\begin{array}{ccc} & & \mathrm{Spin}(n) \\ & \overset{\varepsilon}{\nearrow} & \downarrow \lambda_n \\ \Gamma & \xrightarrow{\pi} & \mathrm{SO}(n) \end{array}$$

- Every crystallographic group is finitely presented. Let

$$\Gamma = \langle S \mid R \rangle,$$

be a presentation of Γ, with finite S and R.

Determining spin structures

For every map $\varepsilon \colon S \to \mathrm{Spin}(n)$ for which $\lambda_n \varepsilon = \pi$ check if it preserves the relations of Γ:

$$r_1 \ldots r_l \in R \overset{?}{\Rightarrow} \varepsilon(r_1) \ldots \varepsilon(r_l) = 1,$$

where $r_1, \ldots, r_l \in S \cup S^{-1}$.

A question

Question

How to determine $\varepsilon(S) \subset \lambda_n^{-1}(G)$?

- $G = \pi(\Gamma) \subset \mathrm{SO}(n)$ – finite group.
- For $n \geq 3$ $\ker \lambda_n = \{\pm 1\}$: for every $x \in \mathrm{Spin}(n), g \in \mathrm{SO}(n)$ we get

$$\lambda_n(x) = g \Rightarrow \lambda_n^{-1}(g) = \{\pm x\}.$$

Remark

From now on we assume $n \geq 3$.

A question

Question

How to determine $\varepsilon(S) \subset \lambda_n^{-1}(G)$?

- $G = \pi(\Gamma) \subset \mathrm{SO}(n)$ – finite group.
- For $n \geq 3$ $\ker \lambda_n = \{\pm 1\}$: for every $x \in \mathrm{Spin}(n), g \in \mathrm{SO}(n)$ we get

$$\lambda_n(x) = g \Rightarrow \lambda_n^{-1}(g) = \{\pm x\}.$$

Remark

From now on we assume $n \geq 3$.

A question

Question

How to determine $\varepsilon(S) \subset \lambda_n^{-1}(G)$?

- $G = \pi(\Gamma) \subset \mathrm{SO}(n)$ – finite group.
- For $n \geq 3 \ker \lambda_n = \{\pm 1\}$: for every $x \in \mathrm{Spin}(n), g \in \mathrm{SO}(n)$ we get

$$\lambda_n(x) = g \Rightarrow \lambda_n^{-1}(g) = \{\pm x\}.$$

Remark

From now on we assume $n \geq 3$.

A question

Question

How to determine $\varepsilon(S) \subset \lambda_n^{-1}(G)$?

- $G = \pi(\Gamma) \subset \mathrm{SO}(n)$ – finite group.
- For $n \geq 3$ $\ker \lambda_n = \{\pm 1\}$: for every $x \in \mathrm{Spin}(n), g \in \mathrm{SO}(n)$ we get

$$\lambda_n(x) = g \Rightarrow \lambda_n^{-1}(g) = \{\pm x\}.$$

Remark

From now on we assume $n \geq 3$.

Isomorphic fundamental groups

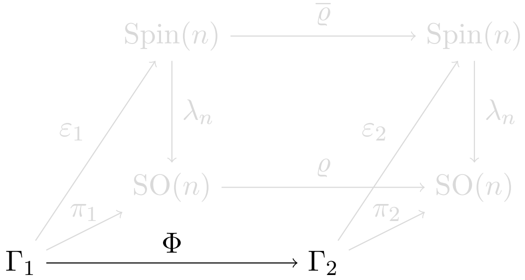
Proposition (Hiss, Szczepański 2008)

Let $\Gamma_1, \Gamma_2 \subset E(n)$ *be isomorphic Bieberbach groups. Then the set of spin structures of* $\mathbb{R}^n/\Gamma_1$ *is in bijection with the set of spin structures of* $\mathbb{R}^n/\Gamma_2$*.*

Isomorphic fundamental groups

Proposition (Hiss, Szczepański 2008)

Let $\Gamma_1, \Gamma_2 \subset E(n)$ *be isomorphic Bieberbach groups. Then the set of spin structures of* $\mathbb{R}^n/\Gamma_1$ *is in bijection with the set of spin structures of* $\mathbb{R}^n/\Gamma_2$.

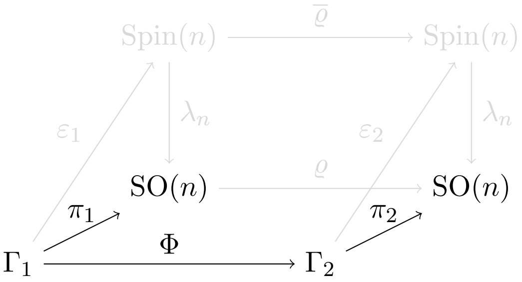


- Φ – the isomorphism: conjugation by $(A, a) \in A(n)$.

Isomorphic fundamental groups

Proposition (Hiss, Szczepański 2008)

Let $\Gamma_1, \Gamma_2 \subset E(n)$ *be isomorphic Bieberbach groups. Then the set of spin structures of* $\mathbb{R}^n/\Gamma_1$ *is in bijection with the set of spin structures of* $\mathbb{R}^n/\Gamma_2$.

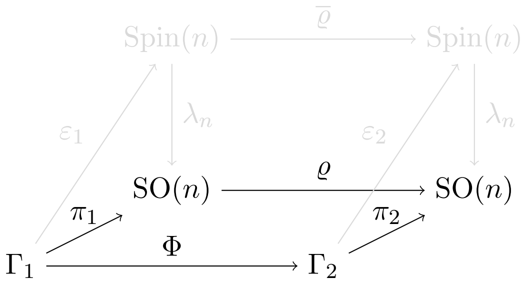


- π_1, π_2 – the projections.

Isomorphic fundamental groups

Proposition (Hiss, Szczepański 2008)

Let $\Gamma_1, \Gamma_2 \subset E(n)$ *be isomorphic Bieberbach groups. Then the set of spin structures of* $\mathbb{R}^n/\Gamma_1$ *is in bijection with the set of spin structures of* $\mathbb{R}^n/\Gamma_2$*.*

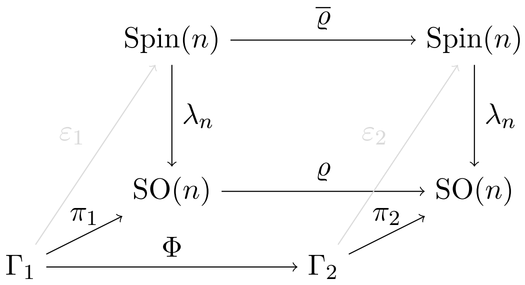


- ϱ – conjugation by $A \in \mathrm{GL}(n, \mathbb{R})$.

Isomorphic fundamental groups

Proposition (Hiss, Szczepański 2008)

Let $\Gamma_1, \Gamma_2 \subset E(n)$ *be isomorphic Bieberbach groups. Then the set of spin structures of* $\mathbb{R}^n/\Gamma_1$ *is in bijection with the set of spin structures of* $\mathbb{R}^n/\Gamma_2$*.*

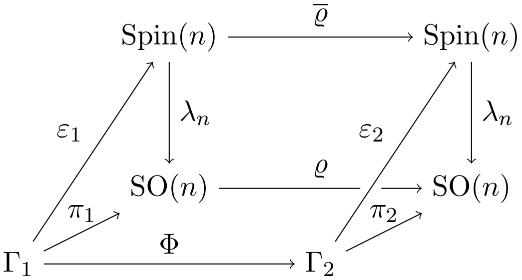


- $\overline{\varrho}$ – conjugation by $\overline{A} \in \mathrm{ML}(n, \mathbb{R})$.

Isomorphic fundamental groups

Proposition (Hiss, Szczepański 2008)

Let $\Gamma_1, \Gamma_2 \subset E(n)$ *be isomorphic Bieberbach groups. Then the set of spin structures of* $\mathbb{R}^n/\Gamma_1$ *is in bijection with the set of spin structures of* $\mathbb{R}^n/\Gamma_2$*.*



$$\varepsilon_2 = \overline{\varrho}\varepsilon_1\Phi^{-1}.$$

Spin structures on flat manifolds

Algebraic condition II

- $X = \mathbb{R}^n/\Gamma$ – orientable flat n-manifold:
$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \stackrel{\pi}{\longrightarrow} G \longrightarrow 1.$$
- $\pi \colon \Gamma \to \mathrm{SO}(n)$.

Corollary

The set of spin structures on X is in bijection with the set of the homomorphisms of the form $\varepsilon \colon \Gamma \to \mathrm{Spin}(n)$ which satisfy $\lambda_n \varepsilon = \varrho\pi$:

$$\begin{array}{ccc} \Gamma & \stackrel{\varepsilon}{\longrightarrow} & \mathrm{Spin}(n) \\ \pi \downarrow & & \downarrow \lambda_n \\ G & \stackrel{\varrho}{\longrightarrow} & \mathrm{SO}(n) \end{array}$$

where $\varrho \colon G \to \mathrm{SO}(n)$ is a representation of G which is $\mathbb{R}$-equivalent to $id \colon G \to G \subset \mathrm{SO}(n)$.

Spin structures on flat manifolds

Algebraic condition II

- $X = \mathbb{R}^n/\Gamma$ – orientable flat n-manifold:

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- $\pi\colon \Gamma \to \mathrm{SO}(n)$.

Corollary

The set of spin structures on X is in bijection with the set of the homomorphisms of the form $\varepsilon\colon \Gamma \to \mathrm{Spin}(n)$ which satisfy $\lambda_n\varepsilon = \varrho\pi$:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\varepsilon} & \mathrm{Spin}(n) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \lambda_n} \\ G & \xrightarrow{\varrho} & \mathrm{SO}(n) \end{array}$$

where $\varrho\colon G \to \mathrm{SO}(n)$ is a representation of G which is $\mathbb{R}$-equivalent to $id\colon G \to G \subset \mathrm{SO}(n)$.

Spin structures on flat manifolds

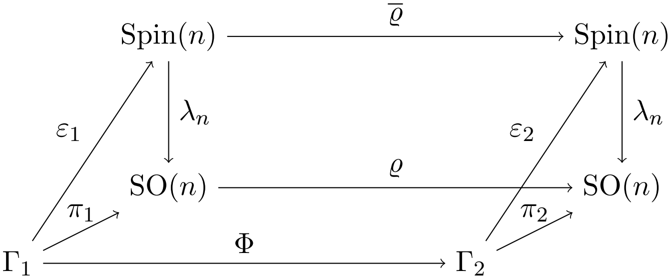
Algebraic condition II

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\varepsilon} & \mathrm{Spin}(n) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \lambda_n} \\ G_1 & \xrightarrow{\varrho} & \mathrm{SO}(n) \end{array}$$

Spin structures on flat manifolds

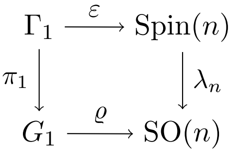
Algebraic condition II

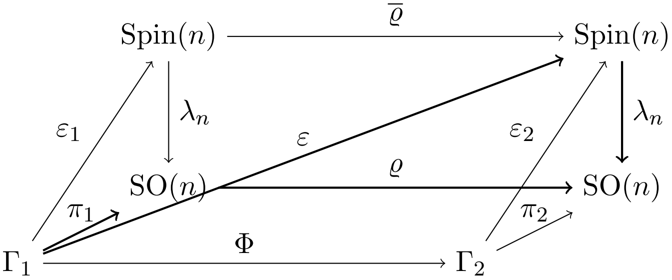
$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\varepsilon} & \mathrm{Spin}(n) \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\lambda_n} \\ G_1 & \xrightarrow{\varrho} & \mathrm{SO}(n) \end{array}$$



Spin structures on flat manifolds

Algebraic condition II





Existence of spin structures on flat manifolds

Necessary and sufficient condition

Lemma

Let Γ be an n-dimensional Bieberbach group with holonomy group G:

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

Let $F \subset G$ be a Sylow 2-subgroup of G. Then $\mathbb{R}^n/\Gamma$ has a spin structure if and only if $\mathbb{R}^n/\pi^{-1}(F)$ has one.

Corollary

It is enough to find a "good" representation $\varrho \colon G \to \mathrm{SO}(n)$ with assumption that G is a 2-group.

Existence of spin structures on flat manifolds

Necessary and sufficient condition

Lemma

Let Γ be an n-dimensional Bieberbach group with holonomy group G:

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

Let $F \subset G$ be a Sylow 2-subgroup of G. Then $\mathbb{R}^n/\Gamma$ has a spin structure if and only if $\mathbb{R}^n/\pi^{-1}(F)$ has one.

Corollary

It is enough to find a "good" representation $\varrho\colon G \to \mathrm{SO}(n)$ with assumption that G is a 2-group.

Existence of spin structures on flat manifolds

Necessary and sufficient condition

Corollary

It is enough to find a "good" representation $\varrho\colon G \to \mathrm{SO}(n)$ with assumption that G is a 2-group.

Theorem (Putrycz, Szczepański 2010)

24 out of the 27 oriented flat 4-manifolds have a spin structure.

Group of "good" matrices

$$\mathrm{O}(n,\mathbb{Z}) := \mathrm{GL}(n,\mathbb{Z}) \cap O(n), \quad \mathrm{SO}(n,\mathbb{Z}) := \mathrm{GL}(n,\mathbb{Z}) \cap \mathrm{SO}(n)$$

- $\mathcal{D} \subset \mathrm{GL}(n,\mathbb{Z})$ – subgroup of diagonal matrices (± 1 on diagonal).
- $P_\sigma \in \mathrm{GL}(n,\mathbb{Z})$ – matrix of a permutation $\sigma \in S_n$.

Lemma

We have the following split exact sequence

$$1 \longrightarrow \mathcal{D} \longrightarrow \mathrm{O}(n,\mathbb{Z}) \longrightarrow S_n \longrightarrow 1$$

with splitting homomorphism defined by

$$\sigma \mapsto P_\sigma.$$

$$\forall_{A \in \mathrm{O}(n,\mathbb{Z})} \exists_{D \in \mathcal{D}} \exists_{\sigma \in S_n} \; A = DP_\sigma$$

Group of "good" matrices

$$\mathrm{O}(n,\mathbb{Z}) := \mathrm{GL}(n,\mathbb{Z}) \cap O(n), \quad \mathrm{SO}(n,\mathbb{Z}) := \mathrm{GL}(n,\mathbb{Z}) \cap \mathrm{SO}(n)$$

- $\mathcal{D} \subset \mathrm{GL}(n,\mathbb{Z})$ – subgroup of diagonal matrices (± 1 on diagonal).
- $P_\sigma \in \mathrm{GL}(n,\mathbb{Z})$ – matrix of a permutation $\sigma \in S_n$.

Lemma

We have the following split exact sequence

$$1 \longrightarrow \mathcal{D} \longrightarrow \mathrm{O}(n,\mathbb{Z}) \longrightarrow S_n \longrightarrow 1$$

with splitting homomorphism defined by

$$\sigma \mapsto P_\sigma.$$

$$\forall_{A\in \mathrm{O}(n,\mathbb{Z})} \exists_{D\in\mathcal{D}} \exists_{\sigma\in S_n}\ A = DP_\sigma$$

Group of "good" matrices

$$\mathrm{O}(n,\mathbb{Z}) := \mathrm{GL}(n,\mathbb{Z}) \cap O(n), \quad \mathrm{SO}(n,\mathbb{Z}) := \mathrm{GL}(n,\mathbb{Z}) \cap \mathrm{SO}(n)$$

- $\mathcal{D} \subset \mathrm{GL}(n,\mathbb{Z})$ – subgroup of diagonal matrices (± 1 on diagonal).
- $P_\sigma \in \mathrm{GL}(n,\mathbb{Z})$ – matrix of a permutation $\sigma \in S_n$.

Lemma

We have the following split exact sequence

$$1 \longrightarrow \mathcal{D} \longrightarrow \mathrm{O}(n,\mathbb{Z}) \longrightarrow S_n \longrightarrow 1$$

with splitting homomorphism defined by

$$\sigma \mapsto P_\sigma.$$

$$\forall_{A \in \mathrm{O}(n,\mathbb{Z})} \exists_{D \in \mathcal{D}} \exists_{\sigma \in S_n} \; A = DP_\sigma$$

Group of "good" matrices

$$\mathrm{O}(n,\mathbb{Z}) := \mathrm{GL}(n,\mathbb{Z}) \cap O(n), \quad \mathrm{SO}(n,\mathbb{Z}) := \mathrm{GL}(n,\mathbb{Z}) \cap \mathrm{SO}(n)$$

- $\mathcal{D} \subset \mathrm{GL}(n,\mathbb{Z})$ – subgroup of diagonal matrices (± 1 on diagonal).
- $P_\sigma \in \mathrm{GL}(n,\mathbb{Z})$ – matrix of a permutation $\sigma \in S_n$.

Lemma

We have the following split exact sequence

$$1 \longrightarrow \mathcal{D} \longrightarrow \mathrm{O}(n,\mathbb{Z}) \longrightarrow S_n \longrightarrow 1$$

with splitting homomorphism defined by

$$\sigma \mapsto P_\sigma.$$

$$\forall_{A \in \mathrm{O}(n,\mathbb{Z})} \exists_{D \in \mathcal{D}} \exists_{\sigma \in S_n} \; A = DP_\sigma$$

Group of "good" matrices

$$\forall_{A \in \mathrm{O}(n,\mathbb{Z})} \exists_{D \in \mathcal{D}} \exists_{\sigma \in S_n} \; A = DP_\sigma$$

- $P_{(i\,j)}$ – matrix of the transposition $(i\,j)$ with -1 instead of 1 in the ith row, where $1 \leq i < j \leq n$.

Corollary

Let $A \in \mathrm{O}(n,\mathbb{Z})$*. Then*

$$A = DP_{(i_1\,j_1)} \dots P_{(i_k\,j_k)},$$

where $D \in \mathcal{D}$*.*

Group of "good" matrices

$$\forall_{A \in \mathrm{O}(n,\mathbb{Z})} \exists_{D \in \mathcal{D}} \exists_{\sigma \in S_n} \; A = DP_\sigma$$

- $P'_{(i\,j)}$ – matrix of the transposition $(i\,j)$ with -1 instead of 1 in the ith row, where $1 \leq i < j \leq n$.

Corollary

Let $A \in \mathrm{O}(n,\mathbb{Z})$. Then

$$A = DP_{(i_1\,j_1)} \dots P_{(i_k\,j_k)},$$

where $D \in \mathcal{D}$.

Group of "good" matrices

$$\forall_{A \in \mathrm{O}(n,\mathbb{Z})} \exists_{D \in \mathcal{D}} \exists_{\sigma \in S_n} \; A = DP_\sigma$$

- $P'_{(i\,j)}$ – matrix of the transposition $(i\,j)$ with -1 instead of 1 in the ith row, where $1 \leq i < j \leq n$.

Corollary

Let $A \in \mathrm{SO}(n,\mathbb{Z})$. *Then*

$$A = DP'_{(i_1\,j_1)} \dots P'_{(i_k\,j_k)},$$

where $D \in \mathcal{D} \cap \mathrm{SO}(n,\mathbb{Z})$.

Preimages of "good" matrices

Lemma

Let $D \in \mathcal{D} \cap \mathrm{SO}(n, \mathbb{Z})$ *has* -1 *in the entries* $i_1 < \ldots < i_m$ *of the diagonal. Then*

$$\lambda_n(e_{i_1} \ldots e_{i_m}) = D.$$

Lemma

$$\forall_{1 \leq i < j \leq n} \ \lambda_n \left(\frac{1 + e_i e_j}{\sqrt{2}} \right) = P'_{(i\,j)}.$$

Preimages of "good" matrices

Lemma

Let $D \in \mathcal{D} \cap \mathrm{SO}(n, \mathbb{Z})$ *has* -1 *in the entries* $i_1 < \ldots < i_m$ *of the diagonal. Then*

$$\lambda_n(e_{i_1} \ldots e_{i_m}) = D.$$

Lemma

$$\forall_{1 \le i < j \le n} \; \lambda_n \left(\frac{1 + e_i e_j}{\sqrt{2}} \right) = P'_{(i\,j)}.$$

Preimages of "good" matrices

Lemma

Let $D \in \mathcal{D} \cap \mathrm{SO}(n, \mathbb{Z})$ *has* -1 *in the entries* $i_1 < \ldots < i_m$ *of the diagonal. Then*

$$\lambda_n(\pm e_{i_1} \ldots e_{i_m}) = D.$$

Lemma

$$\forall_{1 \leq i < j \leq n} \, \lambda_n \left(\pm \frac{1 + e_i e_j}{\sqrt{2}} \right) = P'_{(i\ j)}.$$

Every rational representation of 2-group is "good"

Theorem (Eckmann, Mislin 1979)

Let G be a finite p-group. Then every $\mathbb{Q}$-irreducible representation of G is either induced from a representation of a subgroup of index p or it factors through a representation of a cyclic group of order p.

Corollary A

Every rational representation $\tau\colon G \to \mathrm{GL}(k, \mathbb{Q})$ of 2-group G is equivalent to a representation $\varrho\colon G \to \mathrm{O}(k, \mathbb{Z})$.

Corollary B

Every rational representation $\tau\colon G \to \mathrm{SL}(k, \mathbb{Q})$ of 2-group G is equivalent to a representation $\varrho\colon G \to \mathrm{SO}(k, \mathbb{Z})$.

Every rational representation of 2-group is "good"

Theorem (Eckmann, Mislin 1979)

Let G be a finite p-group. Then every $\mathbb{Q}$-irreducible representation of G is either induced from a representation of a subgroup of index p or it factors through a representation of a cyclic group of order p.

Corollary A

Every rational representation $\tau\colon G \to \mathrm{GL}(k,\mathbb{Q})$ of 2-group G is equivalent to a representation $\varrho\colon G \to \mathrm{O}(k,\mathbb{Z})$.

Corollary B

Every rational representation $\tau\colon G \to \mathrm{SL}(k,\mathbb{Q})$ of 2-group G is equivalent to a representation $\varrho\colon G \to \mathrm{SO}(k,\mathbb{Z})$.

Every rational representation of 2-group is "good"

Corollary A

Every rational representation $\tau\colon G \to \mathrm{GL}(k, \mathbb{Q})$ of 2-group G is equivalent to a representation $\varrho\colon G \to \mathrm{O}(k, \mathbb{Z})$.

Proof.

1. The group $C_2 = \langle c \mid c^2 = 1 \rangle$ has exactly two irreducible representations:

$$c \mapsto 1, \quad c \mapsto -1.$$

□

Every rational representation of 2-group is "good"

Corollary A

Every rational representation $\tau\colon G \to \mathrm{GL}(k, \mathbb{Q})$ of 2-group G is equivalent to a representation $\varrho\colon G \to \mathrm{O}(k, \mathbb{Z})$.

Proof.

2. Assume that the statement is true for every 2-group of order less than $|G|$. Let

$$\tau\colon G \to \mathrm{GL}(k, \mathbb{Q})$$

be an irreducible representation of G.

□

Every rational representation of 2-group is "good"

Corollary A

Every rational representation $\tau\colon G \to \mathrm{GL}(k,\mathbb{Q})$ of 2-group G is equivalent to a representation $\varrho\colon G \to \mathrm{O}(k,\mathbb{Z})$.

Proof.

2. Assume that the statement is true for every 2-group of order less than $|G|$. Let

$$\tau\colon G \to \mathrm{GL}(k,\mathbb{Q})$$

be an irreducible representation of G.

   1. $\tau = \operatorname{ind} \tau_H$, where $H < G, [G:H] = 2$ and $\tau_H\colon H \to \mathrm{GL}(k/2,\mathbb{Q})$:
      - $\tau_H \sim \varrho_H$, where $\varrho_H\colon H \to \mathrm{O}(k/2,\mathbb{Z})$.
      - $\tau = \operatorname{ind}\tau_H \sim \operatorname{ind}\varrho_H$ and $\operatorname{ind}\varrho_H\colon G \to \mathrm{O}(k,\mathbb{Z})$.

□

Every rational representation of 2-group is "good"

Corollary A

Every rational representation $\tau\colon G \to \mathrm{GL}(k,\mathbb{Q})$ of 2-group G is equivalent to a representation $\varrho\colon G \to \mathrm{O}(k,\mathbb{Z})$.

Proof.

2. Assume that the statement is true for every 2-group of order less than $|G|$. Let

$$\tau\colon G \to \mathrm{GL}(k,\mathbb{Q})$$

be an irreducible representation of G.

   1. $\tau = \operatorname{ind}\tau_H$, where $H < G, [G:H] = 2$ and $\tau_H\colon H \to \mathrm{GL}(k/2,\mathbb{Q})$:
      - $\tau_H \sim \varrho_H$, where $\varrho_H\colon H \to \mathrm{O}(k/2,\mathbb{Z})$.
      - $\tau = \operatorname{ind}\tau_H \sim \operatorname{ind}\varrho_H$ and $\operatorname{ind}\varrho_H\colon G \to \mathrm{O}(k,\mathbb{Z})$.

□

Every rational representation of 2-group is "good"

Corollary A

Every rational representation $\tau\colon G \to \mathrm{GL}(k,\mathbb{Q})$ of 2-group G is equivalent to a representation $\varrho\colon G \to \mathrm{O}(k,\mathbb{Z})$.

Proof.

2. Assume that the statement is true for every 2-group of order less than $|G|$. Let

$$\tau\colon G \to \mathrm{GL}(k,\mathbb{Q})$$

be an irreducible representation of G.

   1. $\tau = \operatorname{ind} \tau_H$, where $H < G, [G:H] = 2$ and $\tau_H : H \to \mathrm{GL}(k/2,\mathbb{Q})$:
      - $\tau_H \sim \varrho_H$, where $\varrho_H : H \to \mathrm{O}(k/2,\mathbb{Z})$.
      - $\tau = \operatorname{ind} \tau_H \sim \operatorname{ind} \varrho_H$ and $\operatorname{ind} \varrho_H : G \to \mathrm{O}(k,\mathbb{Z})$.

□

Every rational representation of 2-group is "good"

Corollary A

Every rational representation $\tau\colon G \to \mathrm{GL}(k,\mathbb{Q})$ of 2-group G is equivalent to a representation $\varrho\colon G \to \mathrm{O}(k,\mathbb{Z})$.

Proof.

2. Assume that the statement is true for every 2-group of order less than $|G|$. Let

$$\tau\colon G \to \mathrm{GL}(k,\mathbb{Q})$$

be an irreducible representation of G.

   2. $[G : \ker\tau] = 2$:

$$\tau(G) \subset \{\pm 1\} = \mathrm{O}(1,\mathbb{Z}).$$

□

Every rational representation of 2-group is "good"

Corollary A

Every rational representation $\tau\colon G \to \mathrm{GL}(k, \mathbb{Q})$ of 2-group G is equivalent to a representation $\varrho\colon G \to \mathrm{O}(k, \mathbb{Z})$.

Proof.

2. Assume that the statement is true for every 2-group of order less than $|G|$. Let
$$\tau\colon G \to \mathrm{GL}(k, \mathbb{Q})$$
be an irreducible representation of G.
    2. $[G : \ker \tau] = 2$:
    $$\tau(G) \subset \{\pm 1\} = \mathrm{O}(1, \mathbb{Z}).$$

□

Determining existence of spin structures

Let Γ be a Bieberbach group:

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma' \xrightarrow{\pi} G' \longrightarrow 1,$$

where $G' \subset \mathrm{SO}(n)$, i.e. $\mathbb{R}^n/\Gamma'$ is orientable.

Determining existence of spin structures

Let Γ be a Bieberbach group:

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma' \stackrel{\pi}{\longrightarrow} G' \longrightarrow 1,$$

where $G' \subset \mathrm{SO}(n)$, i.e. $\mathbb{R}^n/\Gamma'$ is orientable.

1. Calculate a Sylow 2-subgroup G of G' and deal with $\Gamma = \pi^{-1}(G) \subset \Gamma'$:

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \stackrel{\pi}{\longrightarrow} G \longrightarrow 1.$$

Determining existence of spin structures

1. $0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1$, G – 2-group.
2. $\varrho\colon G \to \mathrm{SO}(n, \mathbb{Z})$.
3. $G = \langle g_1, \dots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \dots, s$.

2. Determine a representation $\varrho\colon G \to \mathrm{SO}(n, \mathbb{Z})$ equivalent to $id\colon G \to G \subset \mathrm{SO}(n)$.

   Useful:
   - A list of all $\mathbb{Q}$-irreducible integral and orthogonal representations of the group G.
   - Character theory (characteristic zero).

Determining existence of spin structures

1. $0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1$, G – 2-group.
2. $\varrho\colon G \to \mathrm{SO}(n, \mathbb{Z})$.
3. $G = \langle g_1, \ldots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \ldots, s$.

2. Determine a representation $\varrho\colon G \to \mathrm{SO}(n, \mathbb{Z})$ equivalent to $id\colon G \to G \subset \mathrm{SO}(n)$.
   Useful:
   - A list of all $\mathbb{Q}$-irreducible integral and orthogonal representations of the group G.
   - Character theory (characteristic zero).

Determining existence of spin structures

1. $0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1$, G – 2-group.
2. $\varrho \colon G \to \mathrm{SO}(n, \mathbb{Z})$.
3. $G = \langle g_1, \ldots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \ldots, s$.

3. Fix a generating set $\{g_1, \ldots, g_s\}$ of G.
   Using decomposition of $\varrho(g_i)$ determine $x_i \in \mathrm{Spin}(n)$ such that

$$\lambda_n(x_i) = \varrho(g_i)$$

   for every $1 \leq i \leq s$.

Determining existence of spin structures

1. $0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1$, G – 2-group.
2. $\varrho \colon G \to \mathrm{SO}(n, \mathbb{Z})$.
3. $G = \langle g_1, \ldots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \ldots, s$.
4. Determine the integral holonomy representation

$$\varphi \colon G \to \mathrm{GL}(n, \mathbb{Z}).$$

- $\varphi_{i,j} \colon G \to \mathbb{Z}$ – coordinate functions:

$$\forall_{g \in G}\ \varphi(g) = [\varphi_{i,j}(g)].$$

- In CARAT $\Gamma \subset \mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{Q}^n$, $\Gamma \cap \mathbb{Q}^n = \mathbb{Z}^n$.
- ϱ, id_G, φ – $\mathbb{R}$-equivalent.

Determining existence of spin structures

1. $0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1$, G – 2-group.
2. $\varrho \colon G \to \mathrm{SO}(n, \mathbb{Z})$.
3. $G = \langle g_1, \ldots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \ldots, s$.
4. $\varphi \colon G \to \mathrm{GL}(n, \mathbb{Z})$, $\varphi_{i,j} \colon G \to \mathbb{Z}$, $1 \leq i, j \leq n$.

4. Determine the integral holonomy representation

$$\varphi \colon G \to \mathrm{GL}(n, \mathbb{Z}).$$

- $\varphi_{i,j} \colon G \to \mathbb{Z}$ – coordinate functions:

$$\forall_{g \in G}\ \varphi(g) = [\varphi_{i,j}(g)].$$

- In CARAT $\Gamma \subset \mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{Q}^n$, $\Gamma \cap \mathbb{Q}^n = \mathbb{Z}^n$.
- ϱ, id_G, φ – $\mathbb{R}$-equivalent.

Determining existence of spin structures

1. $0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1$, G – 2-group.
2. $\varrho\colon G \to \mathrm{SO}(n,\mathbb{Z})$.
3. $G = \langle g_1, \ldots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \ldots, s$.
4. $\varphi\colon G \to \mathrm{GL}(n,\mathbb{Z})$, $\varphi_{i,j}\colon G \to \mathbb{Z}$, $1 \leq i,j \leq n$.

4. Determine the integral holonomy representation

$$\varphi\colon G \to \mathrm{GL}(n,\mathbb{Z}).$$

- $\varphi_{i,j}\colon G \to \mathbb{Z}$ – coordinate functions:

$$\forall_{g \in G}\ \varphi(g) = [\varphi_{i,j}(g)].$$

- In CARAT $\Gamma \subset \mathrm{GL}(n,\mathbb{Z}) \ltimes \mathbb{Q}^n$, $\Gamma \cap \mathbb{Q}^n = \mathbb{Z}^n$.
- ϱ, id_G, φ – $\mathbb{R}$-equivalent.

Determining existence of spin structures

1. $0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \stackrel{\pi}{\longrightarrow} G \longrightarrow 1$, G – 2-group.
2. $\varrho\colon G \to \mathrm{SO}(n,\mathbb{Z})$.
3. $G = \langle g_1, \ldots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \ldots, s$.
4. $\varphi\colon G \to \mathrm{GL}(n,\mathbb{Z})$, $\varphi_{i,j}\colon G \to \mathbb{Z}$, $1 \leq i,j \leq n$.

4. Determine the integral holonomy representation

$$\varphi\colon G \to \mathrm{GL}(n,\mathbb{Z}).$$

- $\varphi_{i,j}\colon G \to \mathbb{Z}$ – coordinate functions:

$$\forall_{g \in G}\ \varphi(g) = [\varphi_{i,j}(g)].$$

- In CARAT $\Gamma \subset \mathrm{GL}(n,\mathbb{Z}) \ltimes \mathbb{Q}^n$, $\Gamma \cap \mathbb{Q}^n = \mathbb{Z}^n$.
- ϱ, id_G, φ – $\mathbb{R}$-equivalent.

Determining existence of spin structures

2. $\varrho\colon G \to \mathrm{SO}(n,\mathbb{Z})$.
3. $G = \langle g_1, \ldots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \ldots, s$.
4. $\varphi\colon G \to \mathrm{GL}(n,\mathbb{Z})$, $\varphi_{i,j}\colon G \to \mathbb{Z}$, $1 \leq i,j \leq n$.
5. If $\mathbb{Z}^n = \langle a_1, \ldots, a_n \rangle$ and $\gamma_1, \ldots, \gamma_s \in \Gamma$ are such that

$$\pi(\gamma_i) = g_i$$

for $1 \leq i \leq s$ then

$$\Gamma = \langle a_1, \ldots, a_n, \gamma_1, \ldots, \gamma_s \rangle.$$

Determining existence of spin structures

3. $G = \langle g_1, \ldots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \ldots, s$.
4. $\varphi \colon G \to \mathrm{GL}(n, \mathbb{Z})$, $\varphi_{i,j} \colon G \to \mathbb{Z}$, $1 \leq i, j \leq n$.
5. $\Gamma = \langle a_1, \ldots, a_n, \gamma_1, \ldots, \gamma_s \rangle$.

5. If $\varepsilon \colon \Gamma \to \mathrm{Spin}(n)$ – homomorphism with $\lambda_n \varepsilon = \pi$ then

$$\varepsilon(a_i) \in \{\pm 1\} \text{ and } \varepsilon(\gamma_j) \in \{\pm x_i\}$$

for all $1 \leq i \leq n, 1 \leq j \leq s$.

Determining existence of spin structures

3. $G = \langle g_1, \dots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \dots, s$.
4. $\varphi \colon G \to \mathrm{GL}(n, \mathbb{Z})$, $\varphi_{i,j} \colon G \to \mathbb{Z}$, $1 \leq i, j \leq n$.
5. $\Gamma = \langle a_1, \dots, a_n, \gamma_1, \dots, \gamma_s \rangle$.

5. Three types of relations in Γ:
   1. Commutators in $\mathbb{Z}^n$ – automatically satisfied:

$$\varepsilon(\{a_1, \dots, a_n\}) \subset \{\pm 1\}.$$

Determining existence of spin structures

3. $G = \langle g_1, \ldots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \ldots, s$.
4. $\varphi\colon G \to \mathrm{GL}(n, \mathbb{Z})$, $\varphi_{i,j}\colon G \to \mathbb{Z}$, $1 \leq i, j \leq n$.
5. $\Gamma = \langle a_1, \ldots, a_n, \gamma_1, \ldots, \gamma_s \rangle$.

5. Three types of relations in Γ:
   2. The action of G on $\mathbb{Z}^n$. Let $1 \leq i \leq n$ and $1 \leq j \leq s$. In Γ:
   $$a_1^{\varrho_{1i}(g_j)} \cdots a_n^{\varrho_{ni}(g_j)} = \gamma_j a_i \gamma_j^{-1}.$$
   In $\mathrm{Spin}(n)$:
   $$\varepsilon(a_1)^{\varrho_{1i}(g_j)} \cdots \varepsilon(a_n)^{\varrho_{ni}(g_j)} = \varepsilon(\gamma_j)\varepsilon(a_i)\varepsilon(\gamma_j)^{-1}.$$

Determining existence of spin structures

3. $G = \langle g_1, \dots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \dots, s$.
4. $\varphi \colon G \to \mathrm{GL}(n, \mathbb{Z})$, $\varphi_{i,j} \colon G \to \mathbb{Z}$, $1 \leq i, j \leq n$.
5. $\Gamma = \langle a_1, \dots, a_n, \gamma_1, \dots, \gamma_s \rangle$.

5. Three types of relations in Γ:
   2. The action of G on $\mathbb{Z}^n$. Let $1 \leq i \leq n$ and $1 \leq j \leq s$.
      In Γ:
      $$a_1^{\varrho_{1i}(g_j)} \cdots a_n^{\varrho_{ni}(g_j)} = \gamma_j a_i \gamma_j^{-1}.$$
      In $\mathrm{Spin}(n)$:
      $$\varepsilon(a_1)^{\varrho_{1i}(g_j)} \cdots \varepsilon(a_n)^{\varrho_{ni}(g_j)} = \varepsilon(a_i),$$
      since $\varepsilon(a_i) = \pm 1$.

Determining existence of spin structures

3. $G = \langle g_1, \dots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \dots, s$.
4. $\varphi \colon G \to \mathrm{GL}(n, \mathbb{Z})$, $\varphi_{i,j} \colon G \to \mathbb{Z}$, $1 \leq i, j \leq n$.
5. $\Gamma = \langle a_1, \dots, a_n, \gamma_1, \dots, \gamma_s \rangle$.

5. Three types of relations in Γ:
   2. The action of G on $\mathbb{Z}^n$. Let $1 \leq i \leq n$ and $1 \leq j \leq s$.
      In Γ:
      $$a_1^{\varrho_{1i}(g_j)} \cdots a_n^{\varrho_{ni}(g_j)} = \gamma_j a_i \gamma_j^{-1}.$$
      In $\mathrm{Spin}(n)$:
      $$\varepsilon(a_1)^{\varrho_{1i}(g_j)} \cdots \varepsilon(a_n)^{\varrho_{ni}(g_j)} \varepsilon(a_i) = 1,$$
      since $\varepsilon(a_i) = \pm 1$.

Determining existence of spin structures

3. $G = \langle g_1, \ldots, g_s \rangle$, $\lambda_n(x_i) = \varrho(g_i), i = 1, \ldots, s$.
4. $\varphi \colon G \to \mathrm{GL}(n, \mathbb{Z})$, $\varphi_{i,j} \colon G \to \mathbb{Z}$, $1 \leq i, j \leq n$.
5. $\Gamma = \langle a_1, \ldots, a_n, \gamma_1, \ldots, \gamma_s \rangle$.

5. Three types of relations in Γ:
   3. Relations from G.
      In G:
      $$g_{i_1} \cdots g_{i_k} = 1.$$
      In Γ:
      $$\gamma_{i_1} \cdots \gamma_{i_k} = a_1^{\alpha_1} \cdots a_n^{\alpha_n}$$
      for some $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$.
      In $\mathrm{Spin}(n)$:
      $$\varepsilon(\gamma_{i_1}) \cdots \varepsilon(\gamma_{i_k}) = \varepsilon(a_1)^{\alpha_1} \cdots \varepsilon(a_n)^{\alpha_n}.$$

The Bieberbach group

Generators of Γ':

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1/3 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1/2 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$a_i = \begin{bmatrix} I & e_i \\ 0 & 1 \end{bmatrix},$$

where $1 \leq i \leq 5$.

$\Gamma' =$ min.134.1.2.2 in CARAT.

Sylow 2-subgroup

- The short exact sequence for Γ':

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} S_4 \longrightarrow 1.$$

- The sylow 2-subgroup $G = D_8$.
- $\Gamma = \pi^{-1}(G)$ is generated by $a_1, \ldots, a_5$ and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -1 & 1/2 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Sylow 2-subgroup

- The short exact sequence for Γ':

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} S_4 \longrightarrow 1.$$

- The sylow 2-subgroup $G = D_8$.
- $\Gamma = \pi^{-1}(G)$ is generated by $a_1, \ldots, a_5$ and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -1 & 1/2 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Sylow 2-subgroup

- The short exact sequence for Γ':

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} S_4 \longrightarrow 1.$$

- The sylow 2-subgroup $G = D_8$.
- $\Gamma = \pi^{-1}(G)$ is generated by $a_1, \ldots, a_5$ and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -1 & 1/2 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Sylow 2-subgroup

Special orthogonal integral representation

- $\mathrm{Tr}(\pi(A)) = 1, \mathrm{Tr}(\pi(B)) = -1$.
- $a = \pi(A), b = \pi(B)$.
- Character table of G:

	1	a	b	b^2	ab
χ_1	1	1	1	1	1
χ_2	1	−1	1	1	−1
χ_3	1	1	−1	1	−1
χ_4	1	−1	−1	1	1
χ_5	2	0	0	−2	0

- Character of $id\colon G \to G \subset \mathrm{SO}(n)$:

$$\chi_1 + \chi_3 + \chi_4 + \chi_5.$$

Sylow 2-subgroup

Special orthogonal integral representation

- $\mathrm{Tr}(\pi(A)) = 1, \mathrm{Tr}(\pi(B)) = -1$.
- $a = \pi(A), b = \pi(B)$.
- Character table of G:

	1	a	b	b^2	ab
χ_1	1	1	1	1	1
χ_2	1	−1	1	1	−1
χ_3	1	1	−1	1	−1
χ_4	1	−1	−1	1	1
χ_5	2	0	0	−2	0

- Character of $id\colon G \to G \subset \mathrm{SO}(n)$:

$$\chi_1 + \chi_3 + \chi_4 + \chi_5.$$

Sylow 2-subgroup

Special orthogonal integral representation

- $\mathrm{Tr}(\pi(A)) = 1, \mathrm{Tr}(\pi(B)) = -1$.
- $a = \pi(A), b = \pi(B)$.
- Character table of G:

	1	a	b	b^2	ab
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	1	1	-1	1	-1
χ_4	1	-1	-1	1	1
χ_5	2	0	0	-2	0

- Character of $id\colon G \to G \subset \mathrm{SO}(n)$:

$$\chi_1 + \chi_3 + \chi_4 + \chi_5.$$

Sylow 2-subgroup

Special orthogonal integral representation

- $\mathrm{Tr}(\pi(A)) = 1, \mathrm{Tr}(\pi(B)) = -1$.
- $a = \pi(A), b = \pi(B)$.
- Character table of G:

	1	a	b	b^2	ab
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	1	1	-1	1	-1
χ_4	1	-1	-1	1	1
χ_5	2	0	0	-2	0

- Character of $id \colon G \to G \subset \mathrm{SO}(n)$:

$$\chi_1 + \chi_3 + \chi_4 + \chi_5.$$

Sylow 2-subgroup

Special orthogonal integral representation

- Character of $id\colon G \to G \subset \mathrm{SO}(n)$:

$$\chi_1 + \chi_3 + \chi_4 + \chi_5.$$

- $\varphi\colon G \to \mathrm{SO}(5, \mathbb{Z})$ may be defined by

$$a \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, b \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Preimages in $\mathrm{Spin}(5)$

$$\varphi\pi(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\varphi\pi(A) = \lambda_5(\pm e_2 e_4)$$

Preimages in $\mathrm{Spin}(5)$

$$\varphi\pi(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\varphi\pi(A) = \lambda_5(\pm e_2 e_4)$$

Preimages in $\operatorname{Spin}(5)$

$$\varphi\pi(B) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\varphi\pi(B) = \lambda_5\left(\pm e_2e_3e_4e_5\frac{1+e_4e_5}{\sqrt{2}}\right) = \lambda_5\left(\pm e_2e_3\frac{1+e_5e_4}{\sqrt{2}}\right)$$

Preimages in $\mathrm{Spin}(5)$

$$\varphi\pi(B) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\varphi\pi(B) = \lambda_5\left(\pm e_2e_3e_4e_5\frac{1+e_4e_5}{\sqrt{2}}\right) = \lambda_5\left(\pm e_2e_3\frac{1+e_5e_4}{\sqrt{2}}\right)$$

Relations in $\mathrm{Spin}(5)$

The action of G on $\mathbb{Z}^5$

$$\varepsilon(a_1)^{\varrho_{1i}(g_j)} \cdots \varepsilon(a_n)^{\varrho_{ni}(g_j)} \varepsilon(a_i) = 1$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -1 & 1/2 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We get:

$$\begin{cases} \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_4)^2 = 1 \\ \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_5)^2 = 1 \\ \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_4)\varepsilon(a_5)\varepsilon(a_2) = 1 \\ \varepsilon(a_2)\varepsilon(a_5)\varepsilon(a_4) = 1 \end{cases}$$

Relations in $\mathrm{Spin}(5)$

The action of G on $\mathbb{Z}^5$

$$\varepsilon(a_1)^{\varrho_{1i}(g_j)} \cdots \varepsilon(a_n)^{\varrho_{ni}(g_j)} \varepsilon(a_i) = 1$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -1 & 1/2 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We get:

$$\begin{cases} \varepsilon(a_2)\varepsilon(a_3) = 1 \\ \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_5)^2 = 1 \\ \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_4)\varepsilon(a_5)\varepsilon(a_2) = 1 \\ \varepsilon(a_2)\varepsilon(a_5)\varepsilon(a_4) = 1 \end{cases}$$

Relations in $\mathrm{Spin}(5)$

The action of G on $\mathbb{Z}^5$

$$\varepsilon(a_1)^{\varrho_{1i}(g_j)} \cdots \varepsilon(a_n)^{\varrho_{ni}(g_j)} \varepsilon(a_i) = 1$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -1 & 1/2 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We get:

$$\begin{cases} \varepsilon(a_2)\varepsilon(a_3) = 1 \\ \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_5)^2 = 1 \\ \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_4)\varepsilon(a_5)\varepsilon(a_2) = 1 \\ \varepsilon(a_2)\varepsilon(a_5)\varepsilon(a_4) = 1 \end{cases}$$

Relations in $\mathrm{Spin}(5)$

The action of G on $\mathbb{Z}^5$

$$\varepsilon(a_1)^{\varrho_{1i}(g_j)} \cdots \varepsilon(a_n)^{\varrho_{ni}(g_j)} \varepsilon(a_i) = 1$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -1 & 1/2 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We get:

$$\begin{cases} \varepsilon(a_2)\varepsilon(a_3) = 1 \\ \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_5)^2 = 1 \\ \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_4)\varepsilon(a_5)\varepsilon(a_2) = 1 \\ \varepsilon(a_2)\varepsilon(a_5)\varepsilon(a_4) = 1 \end{cases}$$

Relations in $\mathrm{Spin}(5)$

The action of G on $\mathbb{Z}^5$

$$\varepsilon(a_1)^{\varrho_{1i}(g_j)} \cdots \varepsilon(a_n)^{\varrho_{ni}(g_j)} \varepsilon(a_i) = 1$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -1 & 1/2 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We get:

$$\begin{cases} \varepsilon(a_2)\varepsilon(a_3) = 1 \\ \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_5)^2 = 1 \\ \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_4)\varepsilon(a_5)\varepsilon(a_2) = 1 \\ \varepsilon(a_2)\varepsilon(a_4)\varepsilon(a_5) = 1 \end{cases}$$

Relations in $\mathrm{Spin}(5)$

The relations coming from G

$$\varepsilon(\gamma_{i_1})\cdots\varepsilon(\gamma_{i_k}) = \varepsilon(a_1)^{\alpha_1}\cdots\varepsilon(a_n)^{\alpha_n}$$

We have

$$\begin{cases} A^2 &= a_2a_3a_4a_5 \\ B^4 &= a_4a_5^{-1} \\ (AB)^2 &= a_2a_4 \end{cases}$$

We get

$$\begin{cases} \varepsilon(A)^2 &= \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_4)\varepsilon(a_5) \\ \varepsilon(B)^4 &= \varepsilon(a_4)\varepsilon(a_5) \\ (\varepsilon(A)\varepsilon(B))^2 &= \varepsilon(a_2)\varepsilon(a_4) \end{cases}$$

Relations in $\mathrm{Spin}(5)$

Summary

$$\begin{cases} 1 = \varepsilon(a_2)\varepsilon(a_3) \\ 1 = \varepsilon(a_2)\varepsilon(a_4)\varepsilon(a_5) \\ \varepsilon(A)^2 = \varepsilon(a_2)\varepsilon(a_3)\varepsilon(a_4)\varepsilon(a_5) \\ \varepsilon(B)^4 = \varepsilon(a_4)\varepsilon(a_5) \\ (\varepsilon(A)\varepsilon(B))^2 = \varepsilon(a_2)\varepsilon(a_4) \end{cases}$$

We get

$$\begin{cases} \varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_4)\varepsilon(a_5) = \varepsilon(A)^2 = \varepsilon(B)^4 \\ \varepsilon(a_5) = (\varepsilon(A)\varepsilon(B))^2 \end{cases}$$

Relations in $\mathrm{Spin}(5)$

Summary

$$
\begin{cases}
\varepsilon(a_2) & = & \varepsilon(a_3) \\
\varepsilon(a_2) & = & \varepsilon(a_4)\varepsilon(a_5) \\
\varepsilon(A)^2 & = & \varepsilon(a_4)\varepsilon(a_5) \\
\varepsilon(B)^4 & = & \varepsilon(a_4)\varepsilon(a_5) \\
(\varepsilon(A)\varepsilon(B))^2 & = & \varepsilon(a_5)
\end{cases}
$$

We get

$$
\begin{cases}
\varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_4)\varepsilon(a_5) = \varepsilon(A)^2 = \varepsilon(B)^4 \\
\varepsilon(a_5) = (\varepsilon(A)\varepsilon(B))^2
\end{cases}
$$

Relations in $\mathrm{Spin}(5)$

Summary

$$\begin{cases} \varepsilon(a_2) & = & \varepsilon(a_3) \\ \varepsilon(a_2) & = & \varepsilon(a_4)\varepsilon(a_5) \\ \varepsilon(A)^2 & = & \varepsilon(a_4)\varepsilon(a_5) \\ \varepsilon(B)^4 & = & \varepsilon(a_4)\varepsilon(a_5) \\ (\varepsilon(A)\varepsilon(B))^2 & = & \varepsilon(a_5) \end{cases}$$

We get

$$\begin{cases} \varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_4)\varepsilon(a_5) = \varepsilon(A)^2 = \varepsilon(B)^4 \\ \varepsilon(a_5) = (\varepsilon(A)\varepsilon(B))^2 \end{cases}$$

Spin structures on Γ

- $\begin{cases} \varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_4)\varepsilon(a_5) = \varepsilon(A)^2 = \varepsilon(B)^4 \\ \varepsilon(a_5) = (\varepsilon(A)\varepsilon(B))^2 \end{cases}$
- $\varepsilon(A) = \pm e_2 e_4$, $\varepsilon(B) = \pm \frac{e_2 e_3 (1 + e_5 e_4)}{\sqrt{2}}$
- For every possibility we have

$$\varepsilon(A)^2 = \varepsilon(B)^4 = (\varepsilon(A)\varepsilon(B))^2 = -1.$$

- $\varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_5) = -1$, $\varepsilon(a_4) = 1$.
- For a_1, A, B every possibility is good.
- We get 8 spin structures on Γ.

Spin structures on Γ

- $\begin{cases} \varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_4)\varepsilon(a_5) = \varepsilon(A)^2 = \varepsilon(B)^4 \\ \varepsilon(a_5) = (\varepsilon(A)\varepsilon(B))^2 \end{cases}$
- $\varepsilon(A) = \pm e_2 e_4$, $\varepsilon(B) = \pm \frac{e_2 e_3 (1+e_5 e_4)}{\sqrt{2}}$
- For every possibility we have

$$\varepsilon(A)^2 = \varepsilon(B)^4 = (\varepsilon(A)\varepsilon(B))^2 = -1.$$

- $\varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_5) = -1$, $\varepsilon(a_4) = 1$.
- For a_1, A, B every possibility is good.
- We get 8 spin structures on Γ.

Spin structures on Γ

- $\begin{cases} \varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_4)\varepsilon(a_5) = \varepsilon(A)^2 = \varepsilon(B)^4 \\ \varepsilon(a_5) = (\varepsilon(A)\varepsilon(B))^2 \end{cases}$
- $\varepsilon(A) = \pm e_2 e_4$, $\varepsilon(B) = \pm \frac{e_2 e_3 (1 + e_5 e_4)}{\sqrt{2}}$
- For every possibility we have

$$\varepsilon(A)^2 = \varepsilon(B)^4 = (\varepsilon(A)\varepsilon(B))^2 = -1.$$

- $\varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_5) = -1$, $\varepsilon(a_4) = 1$.
- For a_1, A, B every possibility is good.
- We get 8 spin structures on Γ.

Spin structures on Γ

- $\begin{cases} \varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_4)\varepsilon(a_5) = \varepsilon(A)^2 = \varepsilon(B)^4 \\ \varepsilon(a_5) = (\varepsilon(A)\varepsilon(B))^2 \end{cases}$
- $\varepsilon(A) = \pm e_2 e_4$, $\varepsilon(B) = \pm \frac{e_2 e_3 (1 + e_5 e_4)}{\sqrt{2}}$
- For every possibility we have

$$\varepsilon(A)^2 = \varepsilon(B)^4 = (\varepsilon(A)\varepsilon(B))^2 = -1.$$

- $\varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_5) = -1$, $\varepsilon(a_4) = 1$.
- For a_1, A, B every possibility is good.
- We get 8 spin structures on Γ.

Spin structures on Γ

- $\begin{cases} \varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_4)\varepsilon(a_5) = \varepsilon(A)^2 = \varepsilon(B)^4 \\ \varepsilon(a_5) = (\varepsilon(A)\varepsilon(B))^2 \end{cases}$
- $\varepsilon(A) = \pm e_2 e_4$, $\varepsilon(B) = \pm \frac{e_2 e_3 (1 + e_5 e_4)}{\sqrt{2}}$
- For every possibility we have

$$\varepsilon(A)^2 = \varepsilon(B)^4 = (\varepsilon(A)\varepsilon(B))^2 = -1.$$

- $\varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_5) = -1$, $\varepsilon(a_4) = 1$.
- For a_1, A, B every possibility is good.
- We get 8 spin structures on Γ.

Spin structures on Γ

- $\begin{cases} \varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_4)\varepsilon(a_5) = \varepsilon(A)^2 = \varepsilon(B)^4 \\ \varepsilon(a_5) = (\varepsilon(A)\varepsilon(B))^2 \end{cases}$
- $\varepsilon(A) = \pm e_2 e_4$, $\varepsilon(B) = \pm \frac{e_2 e_3 (1+e_5 e_4)}{\sqrt{2}}$
- For every possibility we have

$$\varepsilon(A)^2 = \varepsilon(B)^4 = (\varepsilon(A)\varepsilon(B))^2 = -1.$$

- $\varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_5) = -1$, $\varepsilon(a_4) = 1$.
- For a_1, A, B every possibility is good.
- We get 8 spin structures on Γ.

Spin structures on Γ'

$$\Gamma = \pi^{-1}(G) \subset \Gamma'$$

Corollary

There exists a spin structure on $\mathbb{R}^5/\Gamma'$. Moreover, since

$$H^1(\mathbb{R}^5/\Gamma', \mathbb{Z}_2) = \mathbb{Z}_2^2,$$

there exist exactly four spin structures on the manifold.

Number of spin manifolds

dim	flat mflds	orientable f.m.	spin f.m.
5	1060	174	88
6	38746	3314	760

Relation with holonomy representation

Putrycz, Szczepański 2010

In dimension 4:

1. The existence of a spin structure does not depend on the $\mathbb{Q}$-equivalence class of the integral holonomy representation of an orientable flat manifold.
2. The existence of a spin structure is determined by the $\mathbb{Z}$-equivalence class of the integral holonomy representation of an orientable flat manifold.

Miatello, Podestá 2004

The above does not hold in dimension 6.

The above does not hold in dimension 5.

Relation with holonomy representation

Putrycz, Szczepański 2010

In dimension 4:

1. The existence of a spin structure does not depend on the $\mathbb{Q}$-equivalence class of the integral holonomy representation of an orientable flat manifold.
2. The existence of a spin structure is determined by the $\mathbb{Z}$-equivalence class of the integral holonomy representation of an orientable flat manifold.

Miatello, Podestá 2004

The above does not hold in dimension 6.

The above does not hold in dimension 5.

Relation with holonomy representation

Putrycz, Szczepański 2010

In dimension 4:

1. The existence of a spin structure does not depend on the $\mathbb{Q}$-equivalence class of the integral holonomy representation of an orientable flat manifold.
2. The existence of a spin structure is determined by the $\mathbb{Z}$-equivalence class of the integral holonomy representation of an orientable flat manifold.

Miatello, Podestá 2004

The above does not hold in dimension 6.

The above does not hold in dimension 5.

Thank you!