# Spin structures of flat manifolds of diagonal type

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# Overview

# Bieberbach groups and flat manifolds

- $E(n) = O(n) \ltimes \mathbb{R}^n$ – isometry group of the Euclidean space $\mathbb{R}^n$.
- Discrete and cocompact subgroup $\Gamma \subset E(n)$ – crystallographic group.
- Torsionfree crystallographic $\Gamma \subset E(n)$ – Bieberbach group.
  - $X = \mathbb{R}^n/\Gamma$ – flat manifold (closed connected Riemannian $n$-manifold with zero sectional curvature).
  - $\pi_1(X) \cong \Gamma$.

# Bieberbach theorems

**Theorem (Bieberbach 1911, 1912)**

*Let $\Gamma \subset E(n)$ be a crystallographic group.*

1. *The subgroup $\Gamma \cap (1 \times \mathbb{R}^n)$ of pure translations of $\Gamma$ is free abelian group of rank $n$. Moreover it is maximal abelian normal subgroup of $\Gamma$ of finite index.*
2. *A crystallographic group is isomorphic to $\Gamma$ if and only if it is conjugate to $\Gamma$ in the group $A(n) = \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$.*
3. *There are a finite number of isomorphic classes of crystallographic groups in each dimension.*

# Structure of crystallographic groups

## 1st Bieberbach theorem

- $\Gamma$ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- $\pi \colon \Gamma \to \mathrm{SO}(n)$:

$$\forall_{(A,a)\in\Gamma}\ \pi(A,a) = A.$$

- $G = \pi(\Gamma)$ – finite group – holonomy group of $\Gamma$ ($X$).
- We get a holonomy representation $\varphi \colon G \to \mathrm{GL}(n,\mathbb{Z})$:

$$\varphi_g(z) = \overline{g} z \overline{g}^{-1},$$

  where $z \in \mathbb{Z}^n \subset \Gamma, g \in G, \pi(\overline{g}) = g$.
- $\varphi$ is $\mathbb{R}$-equivalent to $id \colon G \to G \subset \mathrm{SO}(n)$.

# Clifford algebra

## Definition

Let $n \in \mathbb{N}$. The Clifford algebra $C_n$ is a real associative algebra with one, generated by elements $e_1, \ldots, e_n$, which satisfy relations:

$$\forall_{1 \leq i < j \leq n}\; e_i^2 = -1 \text{ and } e_i e_j = -e_j e_i.$$

## Example

$$C_0 = \mathbb{R}, C_1 = \mathbb{C}, C_2 = \mathbb{H}$$

# Three involutions

1. Defined on the generators of the vector space $C_n$:

$$(e_{i_1} \dots e_{i_k})^* = e_{i_k} \dots e_{i_1}.$$

2. Defined on the generators of the algebra $C_n$:

$$e_i' = -e_i.$$

3. The composition of the above two:

$$\forall_{a \in C_n} \; \overline{a} = (a')^*.$$

# Spin group

## Definition

Let $n \in \mathbb{N}$.

$$\mathrm{Spin}(n) := \{x \in C_n \mid x' = x \wedge x\overline{x} = 1\}.$$

# Spin group

## Universal cover of special orthogonal group

### Proposition

*Let* $n \in \mathbb{N}$. *The map* $\lambda_n \colon \operatorname{Spin}(n) \to \operatorname{SO}(n)$, *defined by*

$$\lambda_n(x)v = xv\overline{x},$$

*where* $x \in \operatorname{Spin}(n), v \in \mathbb{R}^n$ *is a continuous group epimorphism.*

For $n \geq 3$:

- $\operatorname{Spin}(n)$ – universal cover of $\operatorname{SO}(n)$.
- $\pi_1(\operatorname{SO}(n)) \cong \mathbb{Z}/2\mathbb{Z}$.
- $\ker \lambda_n = \{\pm 1\}$. We get a central extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow \operatorname{Spin}(n) \xrightarrow{\lambda_n} \operatorname{SO}(n) \longrightarrow 1.$$

# Spin structures on manifolds

- $X$ – orientable closed manifold of dimension $n$.
- $Q$ – principal $\mathrm{SO}(n)$-tangent bundle of $X$.

## Definition

A spin structure on $X$ is a pair $(P, \Lambda)$:

- $P$ – principal $\mathrm{Spin}(n)$-bundle over $X$;
- $\Lambda\colon P \to Q$ is a $2$-fold covering with the commutative diagram:

$$
\begin{array}{ccccc}
P \times \mathrm{Spin}(n) & \longrightarrow & P & & \\
\downarrow{\scriptstyle \Lambda \times \lambda_n} & & {\scriptstyle \Lambda}\downarrow & \searrow & \\
 & & & & X \\
 & & & \nearrow & \\
Q \times \mathrm{SO}(n) & \longrightarrow & Q & &
\end{array}
$$

(The maps in the rows are defined by the action of $\mathrm{Spin}(n)$ and $\mathrm{SO}(n)$ on $P$ and $Q$ respectively.)

**Example**

Every compact oriented $3$ manifold admits a spin structure (it has trivial tangent bundle).

# Stiefel-Whitney classes

## Definition

$E \to X$ – $n$-vector bundle. The *Stiefel-Whitney* classes $w_k(E) \in H^i(X, \mathbb{F}_2)$ are defined by the following axioms:

1. For $f\colon X' \to X$ we have
$$w_k(f^*E) = f^*\big(w_k(E)\big).$$
2. $w_0(E) = 1, w_k(E) = 0$ for $k > n$.
3. $F \to X$ – vector bundle, then
$$w_k(E \oplus F) = \sum_{i=0}^{k} w_i(E) \cup w_{k-i}(F).$$
4. $w_1(\gamma_1) \neq 0$ for the tautological line bundle $\gamma_1$ over $\mathbb{R}P^1$.

# Total Stiefel-Whitney class

$$w(E) := 1 + w_1(E) + \ldots + w_n(E) \in H^*(X, \mathbb{F}_2).$$

# Stiefel-Whitney classes on manifolds

## Definition

$X$ - manifold:

$$w(X) := w(TX).$$

## Proposition

1. $w_1(X) = 0$ *iff* $X$ *is orientable.*
2. $w_1(X) = w_2(X) = 0$ *iff* $X$ *admits a spin structure.*

# Minimal non-spin manifolds

## Definition

Let $X$ be a closed oriented flat manifold, with holonomy group $G$. We say that $X$ is minimal non-spin if it is non-spin and every finite cover with a holonomy group that is a proper subgroup of $G$ has a spin structure.

## Theorem

*For any integer $d \geq 2$ there exists a closed oriented flat manifold $X$ with holonomy group $\mathbb{Z}_2^d$ and $w_2(X) \neq 0$ such that for every finite cover $X'$ of $X$ with holonomy group of rank less than $d$ we have $w(X') = 1$.*

## Theorem

*For any integer $d \geq 2$ there exists a closed oriented flat manifold $M$ with holonomy group $\mathbb{Z}_2^d$ which is minimal non-spin.*

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## Theorem

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## Theorem (A. Gąsior 2013)

*A real Bott manifold with holonomy group of even $\mathbb{Z}_2$-rank has a spin structure if and only if all its finite covers with holonomy group $\mathbb{Z}_2^2$ have a spin structure.*

# Diagonal Bieberbach groups

Let $\Gamma$ be a Bieberbach group defined by a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

## Definition

We call $\Gamma$ *diagonal* or *diagonal type* if and only if the image of the holonomy representation $\rho\colon G \to \mathrm{GL}(n, \mathbb{Z})$ is a group of diagonal matrices.

## Corollary

$G \cong \mathbb{Z}_2^d$ *for some* $d \in \mathbb{N}$*.*

# Action on a torus

- $\Gamma$ – Bieberbach group:

$$\mathbb{R}^n/\Gamma = T^n/G,$$

where $T^n = \mathbb{R}^n/\Lambda, G = \Gamma/\Lambda$ and $\mathbb{Z}^n \cong \Lambda$ – pure translations in $\Gamma$.

- $\Gamma$ – diagonal Bieberbach group: $G \cong \mathbb{Z}_2^d$, $\Lambda = \mathbb{Z}^n$ and

$$\gamma \in \Gamma \Leftrightarrow \gamma = \left( \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}, \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right)$$

where $a_1, \ldots, a_n \in \{\pm 1\}, v_1, \ldots, v_n \in \frac{1}{2}\mathbb{Z}$.

- Let $T^n = (\mathbb{R}/\mathbb{Z})^n$. We get the following action

$$\left( \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}, \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right) \begin{pmatrix} [x_1] \\ \vdots \\ [x_n] \end{pmatrix} = \begin{pmatrix} [a_1 x_1 + v_1] \\ \vdots \\ [a_n x_n + v_n] \end{pmatrix}$$

# Four automorphisms

- Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $g_i\colon S^1 \to S^1$ be given by

$$g_0([z]) = [z],\ \ g_1([z]) = \left[z+\frac{1}{2}\right],\ \ g_2([z]) = [-z],\ \ g_3([z]) = \left[-z+\frac{1}{2}\right]$$

- $\mathcal{D} = \{g_0, g_1, g_2, g_3\} = \langle g_2, g_3 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
- Action of $\mathcal{D}^n$ on $T^n$:

$$(t_1, \ldots, t_n)(z_1, \ldots, t_n) = (t_1 z_1, \ldots, t_n z_n),$$

where $t_i \in \mathcal{D}, z_i \in S^1$ for $i = 1, \ldots, n$.

$$G < \mathcal{D}^n \text{ and } \mathbb{R}^n/\Gamma = T^n/G$$

# The matrix of a diagonal flat manifold

- To a subgroup $\mathbb{Z}_2^d \subset \mathcal{D}^n$ generated by the elements

$$(g_{a_{11}}, \ldots, g_{a_{1n}}),$$
$$\vdots$$
$$(g_{a_{d1}}, \ldots, g_{a_{dn}}),$$

we associate a matrix $A = [a_{ij}]_{d \times n}$.

- Addition of rows in $A$ according to the rule

$$i + j = k \Leftrightarrow g_i g_j = g_k.$$

## Lemma

*Let the matrix $A$ be defined by the subgroup $\mathbb{Z}_2^d \subset \mathcal{D}^n$. Then*

1. (i) *the action of $\mathbb{Z}_2^d$ on $T^n$ is free if and only if there is $1$ in the sum of any distinct collection of rows of $A$,*
2. (ii) *$\mathbb{Z}_2^d$ is the holonomy group of $T^n/\mathbb{Z}_2^d$ if and only if there is either $2$ or $3$ in ethe sum of any distinct collection of rows of $A$.*

# The matrix of a diagonal flat manifold

## Lemma

*Let the matrix $A$ be defined by the subgroup $\mathbb{Z}_2^d \subset \mathcal{D}^n$. Then*

- (i) *the action of $\mathbb{Z}_2^d$ on $T^n$ is free if and only if there is $1$ in the sum of any distinct collection of rows of $A$,*
  *$A$ – free, $A$ – defining matrix of $T^n/\mathbb{Z}_2^d$*
- (ii) *$\mathbb{Z}_2^d$ is the holonomy group of $T^n/\mathbb{Z}_2^d$ if and only if there is either $2$ or $3$ in ethe sum of any distinct collection of rows of $A$.*
  *$A$ – effective*

# Example

## The Hantzsche-Wendt group

$$g_0([z]) = [z],\ \ g_1([z]) = \left[z + \frac{1}{2}\right],\ \ g_2([z]) = [-z],\ \ g_3([z]) = \left[-z + \frac{1}{2}\right]$$

- Let $\Gamma$ be generated by

$$\left(\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}, \begin{bmatrix}1/2\\ 1/2\\ 0\end{bmatrix}\right), \left(\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}, \begin{bmatrix}0\\ 1/2\\ 1/2\end{bmatrix}\right),$$

$$\left(\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1/2\\ 0\\ 1/2\end{bmatrix}\right)$$

# Example

## The Hantzsche-Wendt group

$$g_0([z]) = [z], \;\; g_1([z]) = \left[z + \frac{1}{2}\right], \;\; g_2([z]) = [-z], \;\; g_3([z]) = \left[-z + \frac{1}{2}\right]$$

- Representatives of generators of $G = \Gamma/\mathbb{Z}^3$:

$$\left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} \right) \leftrightarrow \begin{pmatrix} g_1 \\ g_3 \\ g_2 \end{pmatrix}$$

$$\left( \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} \right) \leftrightarrow \begin{pmatrix} g_2 \\ g_1 \\ g_3 \end{pmatrix}$$

# Example

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$$\left( \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} \right) \leftrightarrow \begin{pmatrix} g_2 \\ g_1 \\ g_3 \end{pmatrix}$$

- Let $\mathbb{Z}_2^2 = \langle (g_1, g_3, g_2), (g_2, g_1, g_3) \rangle \subset \mathcal{D}^3$. The defining matrix for

$$\mathbb{R}^3/\Gamma = T^3/\mathbb{Z}_2^2$$

is equal to

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

# The epimorphisms

- We identify $i \leftrightarrow g_i$ for $i = 0, 1, 2, 3$.

## Definition

Define epimorphisms

$$\alpha, \beta : \mathcal{D} \to \mathbb{F}_2$$

be given by

| | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|
| $\alpha$ | $0$ | $1$ | $1$ | $0$ |
| $\beta$ | $0$ | $1$ | $0$ | $1$ |

# The cohomology ring

- $\mathbb{Z}_2^d = \langle b_1, \ldots, b_d \rangle$.
- $\{x_1, \ldots, x_d\} \subset H^1(\mathbb{Z}_2^d; \mathbb{F}_2)$ – basis dual to $\{b_1, \ldots, b_d\}$.

**Theorem**

$$H^*(\mathbb{Z}_2^d; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \ldots, x_d]$$

# Column cocycles

$$A = \begin{bmatrix} A_{11} & \dots & A_{1j} & \dots & A_{1n} \\ \vdots & & \vdots & & \vdots \\ A_{d1} & \dots & A_{dj} & \dots & A_{dn} \end{bmatrix}$$

$$\alpha_j = \sum_{i=1}^{d} \alpha(A_{ij}) x_i, \quad \beta_j = \sum_{i=1}^{d} \beta(A_{ij}) x_i \in \mathbb{F}_2[x_1, \dots, x_d]$$

# Total Stielefel-Whitney class

- $X = T^n/\mathbb{Z}_2^d$ – flat manifold with holonomy $\mathbb{Z}_2^d$.
- $A$ – effective defining matrix for $X$.
- $\Gamma = \pi_1(X)$ fits into the following s.e.s.

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} \mathbb{Z}_2^d \longrightarrow 1$$

## Lemma

$\Gamma$ *acts trivially on* $H^1(\mathbb{Z}^n; \mathbb{F}_2)$.

## 5-term exact sequence

$$0 \to H^1(\mathbb{Z}_2^d; \mathbb{F}_2) \xrightarrow{\pi^*} H^1(\Gamma; \mathbb{F}_2)$$
$$\longrightarrow H^1(\mathbb{Z}^n; \mathbb{F}_2) \xrightarrow{d_2} H^2(\mathbb{Z}_2^d; \mathbb{F}_2) \xrightarrow{\pi^*} H^2(\Gamma; \mathbb{F}_2),$$

# Total Stielefel-Whitney class

## Proposition

1. $w(X) = \pi^*\left(\prod_{j=1}^{n}(1+\alpha_j+\beta_j)\right)$
2. $I_A := \langle \theta_l := \alpha_l\beta_l \mid 1 \leq l \leq n \rangle = \mathrm{Im}(d_2) \subset \ker \pi^*$

## Corollary

*Let* $\mathcal{C}_A = \mathbb{F}_2[x_1, \ldots, x_d]/I_A$. *We get* $\phi\colon \mathcal{C}_A \to H^*(\Gamma; \mathbb{F}_2)$ *given by*

$$[f] = f + I_A \mapsto \pi^*(f).$$

$\phi$ *– mono in gradations* $1$ *and* $2$.

# Total Stielefel-Whitney class

## Definition (SW-class of $A$)

$$w(A) := \left[ \prod_{j=1}^{n} (1 + \alpha_j + \beta_j) \right]$$

## Corollary

$$w(X) = \phi(w(A))$$

# SW-classes of defining matrices

- $A, B$ – matrices of elements of $\mathcal{D}$ with $d$ rows.
- $[A, B]$ – "column concatenation" of $A$ and $B$.

## Lemma

(i) $w([A, B]) = w(A)w(B)$;

(ii) $I_{[A,B]} = I_A + I_B$;

(iii) *if $j$-column of $A$ has only elements $\{0, 2\}$ or $\{0, 3\}$, then $\theta_j^A = 0$.*

# The matrices

$$A_0 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 1 & \dots & 1 \end{bmatrix}_{d \times d-1}, A_1 = \begin{bmatrix} 2 & 2 & \dots & & \\ 3 & & \dots & & \\ & 3 & \dots & & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & \dots & 2 & \\ & & \dots & & 2 \\ & & \dots & 3 & 3 \end{bmatrix}_{d \times \binom{d}{2}},$$

$$A = [A_0, A_1], B = \begin{bmatrix} 2 \\ \vdots \\ 2 \end{bmatrix}_{d \times 1}, C = \begin{bmatrix} 2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{d \times 1}$$

# The matrices

$$E = \begin{cases} [A, B, C, C] & d = 0 \text{ mod } 2 \\ [A, B, B] & d = 1 \text{ mod } 4 \\ A & d = 3 \text{ mod } 4 \end{cases}$$

**Lemma**

*$E$ is free.*

$$F = \begin{cases} E & d \neq 3 \text{ mod } 4 \\ [E, C, C, C, C] & d = 3 \text{ mod } 4 \end{cases}$$

**Lemma**

*$F$ is free and effective.*

# SW-classes of the matrices

## Remark

Denote by $\sigma_k$ the $k$-th elementary symmetric polynomial in variables $x_1, \dots, x_d$.

## Lemma

1. $I_A = \langle x_i^2 + x_j^2 \mid i \neq j \rangle + \langle x_i x_j \mid x_i \neq x_j \rangle$ *and*
$$w(A) = \left[(1 + \sigma_1)^{d-1}\right] \in \mathcal{C}_A;$$
2. $I_B = 0$ *and* $w(B) = [1 + \sigma_1]$.
3. $I_C = 0$ *and* $w(C) = [1 + x_1]$.
4. $I_E = I_A$.

# Minimal non-spin flat manifold

## Proposition

$$w(E) = [1 + x_1^2] \in \mathcal{C}$$

*In particular if $X$ is a flat manifold with definig matrix $E$, then $X$ is oriented, it does not have a spin structure and $w_i(X) = 0$ for all $i > 2$.*

## Proposition

*Let $\mathbb{Z}_2^d \subset \mathcal{D}^n$ be defined by the matrix $E$. Let $G$ be a subgroup of $\mathbb{Z}_2^d$ of rank less than $d$. Then*

$$w(T^n/G) = 1.$$

# Minimal non-spin flat manifold

## Holonomy group of rank $d$

### Theorem

*Suppose $X$ is the flat manifold with defining matrix $F$. Then $X$ is orientable with holonomy group $\mathbb{Z}_2^d$, $w_2(X) \neq 0$ and every finite cover with a holonomy group that is a proper subgroup of $\mathbb{Z}_2^d$ has all vanishing Stiefel-Whitney classes.*

*Thank you!*