Classification of Finite Simple Groups

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Definition

A group G is simple if the only normal subgroups of G are the trivial subgroup and G itself.

Definition

A composition series of a group G is a subnormal series of finite length

$$1 = H_0 \lhd H_1 \lhd \dots \lhd H_n = G$$

with strict inclusions such that each H_i is a maximal strict normal subgroup of H_{i+1}.
Equivalently, a composition series is a subnormal series such that each factor group H_{i+1}/H_i is simple. The factor groups are called composition factors, and the number n is called the composition length.

Remark

Then a group G is the simple product of its composition factors.

Remark

Every finite group has a composition series, but not every infinite group has one.

Corollary

All finite groups are constructed from simple groups.

Question

Is the composition series unique?

Theorem (Jordan-Hölder Theorem 1869-1889)

Let G be a group and assume G has a composition series. Let

$$1 = G_0 \lhd G_1 \lhd \dots \lhd G_n = G,$$

$$1 = H_0 \lhd H_1 \lhd \dots \lhd H_m = G$$

be any two composition series for G. Then $n = m$ and there exists a permutation $\sigma \in S_n$ such that for any $i \in 0, \dots, n-1$, $G_{i+1}/G_i = H_{\sigma(i)+1}/H_{\sigma(i)}$.

Lemma

Let G be a group with A and B – different normal subgroups of G such that G/A and G/B are simple. Then $G/A \cong B/(A \cap B)$ and $G/B \cong A/(A \cap B)$.

Proof.

Suppose that $A \subset B$; then B/A is normal in the simple group G/A. Since $A \neq B$, the quotient is not trivial, and by the assumption that G/B is simple neither is it the whole group. This is a contradiction, so we can assume $A \not\subset B$ and by symmetry $B \not\subset A$. Consider AB – a normal subgroup of G; its image under the quotient map AB/A will be a normal subgroup of G/A. However, from $B \not\subset A$ we have that $AB/A \neq 1$ and so, since G/A is simple, we must have $AB/A = G/A$. Finally, from the second isomorphism theorem we conclude that

$$B/(A \cap B) \cong AB/A = G/A.$$

By symmetry also $A/(A \cap B) \cong G/B$. □

Proof of the Jordan-Hölder Theorem

We use induction over the length of the shortest composition series for G. It is sufficient to show that any composition series is equivalent to a minimal series, and therefore that any two series are equivalent. If G is simple, then it has a unique decomposition series $1 \triangleleft G$. For the inductive case assume that

$$1 = G_0 \lhd G_1 \lhd \ldots \lhd G_{n-1} \lhd G_n = G.$$

is a minimal composition series for G, and

$$1 = H_0 \lhd H_1 \lhd \ldots \lhd H_{m-1} \lhd H_m = G$$

is a composition series.

Suppose that $G_{n-1} = H_{m-1}$; then by induction the series for G_{n-1} will be equivalent to the series for H_{m-1}, and therefore the entire series will be as well.

Proof of the Jordan-Hölder Theorem

Now assume $G_{n-1} \neq H_{m-1}$. Let $K = H_{m-1} \cap G_{n-1}$, which is normal in G. By the lemma we have that $G_{n-1}/K \cong G/H_{m-1}$ and $H_{m-1}/K \cong G/G_{n-1}$ are simple. Let $K_i := K \cap G_i$; then $K_i \triangleleft G_i$ and $K_i \triangleleft K_{i+1}$. Consider the homomorphism $K_{i+1} \to G_{i+1}/G_i$ given by the quotient map. The image is normal and the kernel is K_i, therefore by isomorphism theorems we have that K_{i+1}/K_i is a normal subgroup of G_{i+1}/G_i. Furthermore, since G_{i+1}/G_i is simple, for each K_i, K_{i+1} either $K_i = K_{i+1}$ or the quotient K_{i+1}/K_i is simple. By removing duplicates we get two composition series for G_{n-1}:

$$1 \triangleleft G_1 \triangleleft \ldots \triangleleft G_{n-2} \triangleleft G_{n-1}$$

$$1 \triangleleft K_1 \triangleleft \ldots \triangleleft K_{n-1} \triangleleft G_{n-1}$$

By induction these series are equivalent, and in particular must have the same length, $n-1$, so exactly one of the groups K_{i+1}/K_i is trivial.

Proof of the Jordan-Hölder Theorem

We have already shown that $K_{n-1} \lhd H_{m-1}$ with a simple quotient (note that $K_{n-1} = K$) and therefore we also have the following two composition series:

$$1 \lhd H_1 \lhd \ldots \lhd H_{m-2} \lhd H_{m-1}$$

$$1 \lhd K_1 \lhd \ldots \lhd K_{n-1} \lhd H_{m-1}.$$

Since exactly one of the groups K_{i+1}/K_i is trivial, the lower series is of length n-1, which is less than that of G. Therefore by induction these two series are equivalent with $n-1 = m-1$.

Proof of the Jordan-Hölder Theorem.

It is sufficient to show that the series

$$1 \lhd K_1 \lhd \ldots \lhd K_{n-1} \lhd H_{n-1} \lhd G$$

$$1 \lhd K_1 \lhd \ldots \lhd K_{n-1} \lhd G_{n-1} \lhd G$$

are equivalent. By the lemma $G/G_{n-1} \cong H_{n-1}/K_{n-1}$ and $G/H_{n-1} \cong G_{n-1}/K_{n-1}$ and clearly $K_{i+1}/K_i \cong K_{i+1}/K_i$, therefore this is the case. □

Example (Composition series for $\mathbb{Z}_{12}$)

- $$1 = \mathbb{Z}_1 \lhd \mathbb{Z}_2 \lhd \mathbb{Z}_4 \lhd \mathbb{Z}_{12}$$
  Composition factors: $\mathbb{Z}_{12}/\mathbb{Z}_4 \cong \mathbb{Z}_3$, $\mathbb{Z}_4/\mathbb{Z}_2 = \mathbb{Z}_2$, $\mathbb{Z}_2/\mathbb{Z}_1 \cong \mathbb{Z}_2$.
- $$1 = \mathbb{Z}_1 \lhd \mathbb{Z}_2 \lhd \mathbb{Z}_6 \lhd \mathbb{Z}_{12}$$
  Composition factors: $\mathbb{Z}_{12}/\mathbb{Z}_6 \cong \mathbb{Z}_2$, $\mathbb{Z}_6/\mathbb{Z}_2 = \mathbb{Z}_3$, $\mathbb{Z}_2/\mathbb{Z}_1 \cong \mathbb{Z}_2$.
- $$1 = \mathbb{Z}_1 \lhd \mathbb{Z}_3 \lhd \mathbb{Z}_6 \lhd \mathbb{Z}_{12}$$
  Composition factors: $\mathbb{Z}_{12}/\mathbb{Z}_6 \cong \mathbb{Z}_2$, $\mathbb{Z}_6/\mathbb{Z}_3 = \mathbb{Z}_2$, $\mathbb{Z}_3/\mathbb{Z}_1 \cong \mathbb{Z}_3$.

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Theorem (The Classification Theorem)

Let G be a finite simple group. Then G is either

(a) *a cyclic group of prime order;*

(b) *an alternating group of degree $n \geq 5$;*

(c) *a finite simple group of Lie type;*

(d) *one of 26 sporadic finite simple groups: the five Mathieu groups, the four Janko groups, the three Conway groups, the three Fischer groups, HS, Mc, Suz, Ru, He, Ly, ON, HN, Th, BM and M.*

Dawn of the project

- Hölder (1892): "It would be of the greatest interest if it were possible to give an overview of the entire collection of finite simple groups".
- Cole (1892-1893): determined all simple groups of orders up to 660, discovering a new group $SL(2, 8)$.
- Miller and Ling (1900): up to 2001.
- Only available tools: Sylow's Theorems and the Pigeonhole Principle.

Alternate strategy

- Cole and Glover (1893): structure of a finite group depends more on the shape of prime factorisation of $|G|$ than actual nature of prime factors.
- Frobenius (1893), Burnside (1895): importance of the smallest prime divisor p of $|G|$ and the structure of a Sylow p-subgroup.
- Dedekind (April 6, 1896) invited Frobenius to consider the problem of factoring the group determinant of a finite nonabelian group.
- Frobenius determinant theorem (1896): the birth of the theory of group characters.

- Burnside (1900): applied new theory to show that if G is a nonabelian simple group of odd order, then $|G| > 40000$, $|G|$ must have at least seven prime factors, and G can have no proper subgroup of index less than 101.

Theory building

- Hall (1928,1937): series of papers on finite solvable groups, generalisations of Sylow's Theorems.
- Wielandt, Hall, Kegel and others (1950's): connection between solvability and factorisations.
- Zassenhaus (1937): focused on the architectural structure of groups in terms of normal subgroups and factor groups; the Schur-Zassenhaus theorem.

Odd Order Conjecture (Miller, Burnside)

Every finite group of odd order is solvable.

Schreier Conjecture

If N is a nonabelian finite simple group, then $Aut(N)/N$ is a solvable group.

- Both these conjectures turn out to be true, but very deep.
- The only known proof of the Schreier Conjecture is as Corollary of the Classification Theorem.
- The proof of the Odd Order Conjecture by Feit and Thompson yielded the unrestricted Schur-Zassenhaus Theorem; no elementary proof is known.

Definition (The Fitting sugroup)

The Fitting subgroup $F(G)$ of a finite group G is the join of all normal nilpotent subgroups of G.

Theorem (Fitting's Theorem (1938, edited posthumously by Zassenhaus))

Let G be a finite solvable group. Then $C_G(F(G)) \leq F(G)$.

Remark

This is false for general finite groups; for example, if G is a nonabelian simple group, $F(G) = 1$.

Definition (Weakly closed subgroup)

Let H be a subgroup of the group G. A subgroup W of H is weakly closed in H (with respect to G) if $W^g \leq H$ implies $W^g = W$ for all $g \in G$; i.e., W is the unique member of its G-conjugacy class which is contained in H.

- Grün (1936): if the center $Z(P)$ is a weakly closed subgroup of P, then G has an abelian p-quotient if and only if $N_G(Z(P))$ does.
- Zassenhaus (1930's): extension of Jordan and Frobenius' work on transitive permutation groups.
- Brauer (1930's): investigation of modular representations of finite groups.

Classification begins in earnest

- Zassenhaus (1947) hoped to linearise the problem by identifying all simple groups as groups of automorphisms of some linear structure, perhaps a finite Lie algebra.
- Chevalley (1955) found a uniform method to construct finite analogues of the simple complex Lie groups.
- Lie theoretic context for all of the known simple groups except for the alternating groups and the five Mathieu groups; new finite simple groups, unified context for the study of their subgroups and presentations; no obvious strategy for classification.

- Brauer, Fowler (1948–): CA-groups ($=$ centraliser of ever non-identity element is abelian) of even order.
- Suzuki (1950): characterisation of $PGL(2, q)$, q odd, in terms of partitions.
- Brauer, Suzuki, Wall (1953): characterisation of $PSL(2, q)$,The Brauer-Suzuki-Wall Theorem; special case – Burnside in 1899, rediscovered 1970 by Feit.

- Brauer-Fowler Theorem (1955): bound on the order of a finite simple group of even order given the order of one of its involution centralisers.
- For any finite group H the determination of all finite simple groups with an involution centraliser isomorphic to H is a finite problem.
- Two-step strategy for proving the Classification Theorem:
  1. Determine all possible structures for an involution centraliser in a finite simple group.
  2. For each possible structure, determine all finite simple groups with such an involution centraliser.
- Brauer had proved some sample cases for Step 2; no one had an idea how to do Step 1.

- Suzuki (1957): nonexistence of nonabelian simple *CA*-groups of odd order.
- First breakthrough in the direction of the Miller-Burnside Odd Order Conjecture.
- Nevertheless, difficulties still seemed insurmountable.

Enter John Thompson

Theorem (Thompson's Thesis 1959/60)

Let G be a finite group admitting an automorphism α of prime order with $C_G(\alpha) = 1$. Then G is a nilpotent group.

- If G is nilpotent, then for every prime p dividing $|G|$, G has a normal subgroup P of index p which is α-invariant. Then $Z(P)$ cannot be weakly closed in P. This led Thompson to study weak closures of abelian subgroups of P.
- Discovery of the J-subgroup and the Thompson factorisation theorems.

Definition

Let $A \leq H \leq G$. The weak closure of A in H with respect to G is

$$W = \langle A^g : A^g \leq H \rangle.$$

Equivalently, W is the smallest subgroup of H containing A and weakly closed in H (with respect to G).

Definition (Thompson subgroup)

Let P be a finite p-group and let d be the maximum rank of an elementary abelian subgroup of P. Let $\mathcal{A}(P)$ denote the set of all elementary subgroups of P of rank d. Then the Thompson subgroup $J(P)$ is

$$J(P) = \langle A : A \in \mathcal{A}(P) \rangle.$$

Let H be a finite solvable group whose Fitting subgroup F is a p-group. For R – any p-group denote by $\Omega_1(R)$ the subgroup generated by the elements of order p in R.

Thompson Factorisation

$$H = CN_H(J(P)) = C_H(\Omega_1(Z(P)))N_H(J(P))$$

When hypotheses such as solvability and odd order are dropped, the analysis becomes much more complicated, but the fundamental philosophy remains the same.

Definition

A finite group is of (local) characteristic p-type if the following condition is satisfied by every p-local subgroup H of G: Let F be the largest normal p-subgroup of H. Then $C_H(F) \leq F$.

- When G is a group of characteristic p-type, Thompson's analysis may be undertaken.
- Glauberman (around 1967): the ZJ-Theorem; easier approach in the context of groups of odd order.
- Stellmacher (1996): an analogue of the ZJ-Theorem for groups of order prime to 3.
- In general context: Thompson's factorisation.

- Hall and Thompson (1959): extended Suzuki's theorem on *CA*-groups of odd order to the nilpotent centraliser case; Feit improved character theory.
- Feit and Thompson: collaboration on groups of odd order.

Theorem (The Odd Order Theorem, 1963)

All finite groups of odd order are solvable.

Corollary

A finite simple group is either prime cyclic or of even order.

The paper is 225 pages long.

Back to the prime 2

- The only known models were the Odd Order Paper and Thompson's evolving work on minimal simple groups of even order.
- Dichotomy between groups of p-rank at most 2 and those of p-rank at least 3 → importance of groups of 2-rank 2 as a separate problem.

Definition

The p-rank of a group G is the largest $n \in \mathbb{Z}$ such that G has an elementary abelian subgroup of order p^n.

- Alperin: a 2-group of 2-rank 2 which is a candidate to be a Sylow 2-subgroup must be one of dihedral, semidihedral, wreathed or homocyclic abelian.

- Alperin, Brauer, Gorenstein (1969): classified most simple groups of 2-rank at most 2.
- 2-rank at least 3: Signaliser functor analysis.
- Bender's Strongly Embedded Theorem (1971).
- Aschbacher (1973): 2-Uniqueness Theorem, needed for the Signalizer Functor Method.

- Janko (1974): study of involution centralisers – new sporadic simple group J_4.
- 20 more sporadic simple groups.
- Walter (1969): reduction of the problem of groups with abelian Sylow 2-subgroups to the specific centraliser of involution problem which had been studied by Thompson, Janko and others.
- Bender (1970): generalised Fitting subgroup.
- Gorenstein, Walter (1975): L-Balance Theorem.
- Comparison of the centralisers of two different commuting involutions in a finite group G.
- Goldschmidt : candidate signalizer functor, modified and implemented by Aschbacher in his characterisations of simple groups of Lie type.

Gorenstein's Classification Programme (1972)

- Step I – "Small odd type" case, completed by Gorenstein and Harada.
- Step II – Signaliser analysis for the "large odd type".
- Steps IV, VI, VII and VIII – various aspects of the Odd Type Case: final identification problem for the groups of Lie type in odd characteristic, the alternating groups and most sporadic simple groups.
- Steps III and V – Bounding the number of quasisimple components in the centraliser of some involution.

A first version of such a bounding theorem was proved by Powell and Thwaites; shortly thereafter, an optimal theorem was obtained by Aschbacher (1973).

The Monster

Fischer (1973) and Griess (1976) predicted the existence of the Monster Group, also known as the Fischer-Griess Monster or the Friendly Giant,the largest sporadic simple group. The order of the Monster is

$$\begin{aligned} &2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \\ &= 808,017,424,794,512,875,886,459,904,961,710,757,005, \\ &\qquad 754,368,000,000,000 \\ &\approx 8 \cdot 10^{53}. \end{aligned}$$

- The Monster group contains the double cover of the Baby Monster group as a centraliser of an involution.
- The Monster group contains 20 sporadic groups (including itself) as subquotients.
- The character table of the Monster is a 194-by-194 array.

Gorenstein's Classification Programme

- Step IX – Thin groups (Aschbacher)
- Step X – Groups with a stronly p-embedded (2-local) subgroup, p odd (Aschbacher)
- Steps XI and XV – The signaliser functor method and component theorem for odd primes (Gorenstein-Lyons)
- Step XII – Groups of characteristic $2, p$-type (Timmesfield et al.)
- Step XIII – Quasinthin groups (Mason, Aschbacher-Smith)
- Step XIV – Groups with $e(G) = 3$ (Aschbacher)
- Step XVI – Final characterisation of the simple groups of characteristic 2-type (Gilman-Griess)

Definition

Let G be a finite group. For H a local 2-subgroup of G, denote by $e(H)$ the maximum rank of an abelian subgroup of H of odd prime-power order. Then let $e(G)$ denote the maximum value of $e(H)$ as H ranges over all 2-local subgroups of G.

By the Odd Order Theorem and a theorem of Frobenius, if G is nonsolvable, then $e(G) \geq 1$.

Definition

When $e(G) = 1$, we call G a thin group.

- Canonical examples: groups of Lie type in characteristic 2.
- $PSL(2, 2^n)$, $PSU(3, 2^n)$, $Sz(2^n)$, $PSL(3, 4)$.
- Aschbacher: Thin Group Theorem.
- Aschbacher, Foote et al.: Global $C(G, T)$ Theorem.
- Unresolved: the cases $e(G) = 2$ (the quasithin case) and $e(G) = 3$, claimed by Mason and Aschbacher, respectively.
- Gap: existence and uniqueness of some of the sporadic simple groups, notable the Monster.
- Griess (January 1980): computer-free construction of the Monster, as group of automorphisms of a non-associative commutative algebra of dimension 196 884.

Theorem (The Classification Theorem)

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- August 1980/February 1981: Gorenstein asserts completion.
- 1989: Aschbacher noticed that Mason's 800-page manuscript on quasithin groups was incomplete in various ways.
- 1992: Aschbacher prepared a manuscript treating the remaining cases; still unpublished.
- 1996: Aschbacher and Smith took on the task of proving and publishing a proof of the Quasithin Theorem.
- Published in 2004.
- Finished! :)

Based on

A brief history of the classification of finite simple groups

by Ronald Solomon

BULLETIN (New Series) OF THE AMERICAN MATHEMATICAL SOCIETY Volume 38, Number 3, Pages 315–352.

Thank you for attention.