Representations and characters

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Definition

A representation of a group G is a homomorphism

$$\rho : G \to GL(V)$$

where V is a vector space.

Definition

A representation is said to be faithful if it is injective.

Definition

The trivial representation is the representation $\rho \equiv 1$.

Example (Standard representation of S_3)

Let $V = \langle v_1, v_2, v_3 \rangle / \langle v_1 + v_2 + v_3 \rangle$ and let $\rho : S_3 \to GL(V)$ be defined as $\rho(\sigma)v_i = v_{\sigma(i)}$ for all $\sigma \in S_3$.Then ρ maps

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, (12) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(13) \mapsto \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, (23) \mapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$(123) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, (132) \mapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

Definition

An FG-module is a vector space V over a field F together with group action. That is, for all $g \in G$, $\alpha \in F$ and $u, v \in V$ $g \cdot v$ is defined and satisfies

1. $g \cdot (\alpha v) = \alpha(g \cdot v)$;
2. $g \cdot (u + v) = g \cdot u + g \cdot v$.

We now let $gv := \rho(g)v$, and we can say that ρ gives V the structure of an FG-module.

Remark

For shorthand, we call the FG-module the representation of G; that is, if $\rho : G \to GL(V)$, we call V the representation of G instead of ρ.

Definition

Let V be a representation. W is a subrepresentation of V if W is a subspace of V that is invariant under G, that is

$$\forall_{w \in W} \forall_{g \in G} gw \in W.$$

Definition

A representation V is said to be irreducible if the only subrepresentations of V are $\{0\}$ and V itself.

Definition

Let V, W be representations. A function $\phi : V \to W$ is called a G-linear map if ϕ is a linear transformation and it satisfies

$$\forall_{g \in G} \forall_{v \in V} \phi(gv) = g\phi(v).$$

Proposition

Let V, W be representations and $\phi : V \to W$ – a G-linear map. Then $\ker\phi$ is a subrepresentation of V and $\mathrm{im}\phi$ is a subrepresentation of W.

Proof.

Since ϕ is a linear transformation, it follows that $\ker\phi$ is a subspace of V and $\mathrm{im}\phi$ is a subspace of W.
Let $u \in \ker\phi$ and $g \in G$. Then $\phi(gu) = g\phi(u) = g \cdot 0 = 0$, and so $gu \in \ker\phi$ and $\ker\phi$ is invariant under G.
Now let $w \in \mathrm{im}\phi$ and $g \in G$; there exists $v \in V$ such that $\phi(v) = w$. Then $\phi(gv) = g\phi(v) = gw \in W$. $\mathrm{im}\phi$ is invariant under G. □

Definition

The representations V, W are isomorphic if there exists a G-linear map $\phi : V \to W$ that is invertible, i.e. ϕ is a representation isomorphism. We write $V \cong W$.

Definition

Let FG be a vector space over the field F with a basis $g_1, \dots, g_n \in G$, where $g_1, \dots, g_n$ are all the elements of a finite group G.Then for all $v \in V$

$$v = \sum_{i=1}^{n} \lambda_i g_i,\ \lambda_i \in F$$

or equivalently,

$$v = \sum_{g \in G} \lambda_g g,\ \lambda_g \in F.$$

Then FG is a group algebra of G.

Definition

Let FG be a group algebra of a finite group G. The vector space FG, together with group algebra, is called the regular FG-module or the regular representation of G.

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Theorem (Maschke's Theorem)

Let V be a representation of a finite group G. Then if there exists a subrepresentation U of V, there must also be a subrepresentation W of V such that $V = U \oplus W$.

Proof.

Let U be a subrepresentation of V. We choose any complementary subspace W_0 of V such that $V = U \oplus W_0$. For all $v \in V$, $v = u + w$ for unique $u \in U$, $w \in W_0$. Let $\pi_0 : V \to U$ be the projection of V given by $v \mapsto u$. We then average π_0 over G to create a G-linear map. Let

$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi_0(gv).$$

Proof.

We need to show that $\pi(v)$ is a G-linear map. Let $g, h, x \in G$ such that $h = gx$. Then for all $v \in V$, $\pi(xv) = \frac{1}{|G|}\sum_{g\in G} g^{-1}\pi_0(gxv) = \frac{1}{|G|}\sum_{h\in G} xh^{-1}\pi_0(hv) = x\left(\frac{1}{|G|}\sum_{h\in G} h^{-1}\pi_0(hv)\right) = x\pi(v)$. Hence π is a G-linear map. Note that for all $u \in U$, $\pi(u) = u$ since $\pi_0(gu) = gu$. This means that $\mathrm{im}\pi = U$. $\ker\pi$ is also a subrepresentation of V; let $\ker\pi = W$. Then $V = U \oplus W$. □

Definition

A representation V is completely reducible if V can be written as the direct sum of irreducible representations.

Corollary

Every representation of a finite group is completely reducible.

Proof.

By induction. Let V be a representation. If $\dim V = 0$ or 1, V is irreducible and the result holds. Suppose V is not irreducible. Then V has a subrepresention U aside from $\{0\}$ and V. By Maschke's Theorem there exists another subrepresentation W such that $V = U \oplus W$. It is clear that $\dim U$ and $\dim W$ are less than $\dim V$. Then by induction we can find subrepresentations of U, W such that $U = U_1 \oplus \ldots \oplus U_r$, $W = W_1 \oplus \ldots \oplus W_s$, where U_i, W_j – irreducible. Then

$$V = U_1 \oplus \ldots \oplus U_r \oplus W_1 \oplus \ldots \oplus W_s.$$

□

Remark

This works for any field of characteristic prime to the order of G.

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Theorem (Schur's Lemma)

Let V, W be irreducible representations of a finite group G and let $\phi : V \to W$ be a G-linear map. Then

1. *Either ϕ is an isomorphism or $\phi = 0$.*
2. *If $V = W$, then $\phi = \lambda I_V$ for some $\lambda \in \mathbb{C}$.*

Proof.

1. Suppose $\phi \neq 0$. Then $\ker \phi \neq V$. $\ker \phi$ is a subrepresentation of V. Since V is irreducible and $\ker \phi \neq V$, we deduce that $\ker \phi = \{0\}$. Likewise, $\operatorname{im} \phi \neq \{0\}$ and W is irreducible, so $\operatorname{im} \phi = W$. Hence ϕ is a bijective G-linear map, or an isomorphism.

Proof.

2. Let $V = W$. Since $\mathbb{C}$ is algebraically closed, ϕ has an eigenvalue $\lambda \in \mathbb{C}$ for which $\ker(\phi - \lambda I_V) \neq \{0\}$. V is irreducible, so this implies $\ker(\phi - \lambda I_V) = V$. Then λ is an eigenvalue for all $v \in V$. Thus $\phi = \lambda I_V$.

□

Corollary

Let V be a representation of G. Then V is irreducible if and only if every automoprhism A of V that is also a G-linear map takes the form $A = \lambda I_V$ for some $\lambda \in \mathbb{C}$.

Theorem

Every irreducible representation of a finite abelian group has dimension 1.

Proof.

Let G be a finite abelian group and let V be an irreducible representation of G. For all $g, h \in G$ and $v \in V$,

$$h(gv) = (hg)v = (gh)v = g(hv).$$

This shows that every $g \in G$ is a G-linear map $g : V \to V$. By Schur's Lemma, g takes the form $g = \lambda I_V$ for some $\lambda \in \mathbb{C}$. If W is a subspace of V, then for all $g \in G$, $w \in W$, $gw = \lambda w \in W$. But this means that every subspace of V is invariant under G. Since V is irreducible, this implies $\dim V = 1$. □

Corollary

Let $\mathbb{C}G$ be the regular representation of G. By Corollary, $\mathbb{C}G$ can be written as $\mathbb{C}G = U_1 \oplus \ldots \oplus U_r$, $U_1, \ldots, U_r$–irreducible. Let W be any irreducible representation of G. Then $W \cong U_i$ for some i.

Corollary

Every finite group has only finitely many non-isomorphic irreducible representations.

Theorem

Every representation V of a finite group G has a decomposition $V \cong U_1^{\oplus a_1} \oplus \ldots \oplus U_r^{\oplus a_r}$, where $U_1, \ldots, U_r$ – non-isomorphic irreducible representations and each U_i has a_i multiplicities.

Proof.

Let V be the direct sum of irreducible representations $W_1 \oplus ... \oplus W_s$. Take W_1 and all $W_{j_1}, ..., W_{j_k}$ such that $W_1 \cong W_{j_1} \cong ... \cong W_{j_k}$. Let $W_1 = U_1$. Then

$$W_1 \oplus W_{j_1} \oplus ... \oplus W_{j_k} \cong U_1^{\oplus a_1},$$

where $a_1 = k + 1$. Now consider $S = \{W_2, ..., Ws\} \setminus \{W_1, W_{j_1}, ..., W_{j_k}\}$. Take any $W_l \in S$ and find all W_i that are isomorphic to W_l. Allow $W_l = U_2$.Then we can find a_2 such that

$$\bigoplus(\text{all } W_i \in S \text{ isomorphic to } W_l) \cong U_2^{\oplus a_2}.$$

Proof.

We repeat this process until we arrive at

$$V \cong U_1^{\oplus a_1} \oplus \ldots \oplus U_r^{\oplus a_r}$$

where $U_1, \ldots, U_r$ are non-isomorphic and irreducible. □

Remark

$U_1, \ldots, U_r$ are called the complete set of non-isomorphic irreducible representations of G.

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Definition

The character of the representation $\rho : G \to GL(V)$ is the function $\chi : G \to \mathbb{C}$ defined as

$$\chi(g) = \mathrm{tr}(\rho(g)).$$

Properties

- χ does not depend on the choice of basis for V.
- χ is class function, i.e. it is constant along conjugacy classes.
- χ is irreducible if and only if V is irreducible.

Example (Standard representation of S_3)

Let $V = \langle v_1, v_2, v_3\rangle / \langle v_1 + v_2 + v_3\rangle$ and let $\rho : S_3 \to GL(V)$ be defined as $\rho(\sigma)v_i = v_{\sigma(i)}$ for all $\sigma \in S_3$. Then ρ maps

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, (12) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
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and the character of this representation is

$$\chi(1) = 2$$
$$\chi((12)) = \chi((13)) = \chi((23)) = 0$$
$$\chi((123)) = \chi((132)) = -1$$

Proposition

Let V be a representation of G written as $V = U_1 \oplus \ldots \oplus U_r$. Then $\chi_V = \chi_{U_1} + \ldots + \chi_{U_r}$.

Corollary

Suppose now that V is isomorphic to a direct sum of irreducible representations, $V \cong U_1 \oplus \ldots \oplus U_r$. Then it still follows that $\chi_V = \chi_{U_1} + \ldots + \chi_{U_r}$.

Definition

A class function on G is a function $\phi : G \to \mathbb{C}$ such that for $g, h -$ conjugate in G, $\phi(g) = \phi(h)$.

Definition

$\mathbb{C}_{class}(G)$ – set of all class functions $G \to \mathbb{C}$. Then we can define a Hermitian inner product $\langle , \rangle$ on $\mathbb{C}_{class}(G)$ in the following way: For $\chi, \psi \in \mathbb{C}_{class}(G)$, let

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

Then $\langle \chi, \psi \rangle = \overline{\langle \psi, \chi \rangle}$.

Theorem

The irreducible characters of G are orthonormal, that is, if V, W – irreducible representations with characters χ, ψ, then

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W \end{cases}$$

We can now deduce some basic facts about characters from what we already know about inner products. Let V be a representation written as $V \cong U_1^{\oplus a_1} \oplus \ldots \oplus U_r^{\oplus a_r}$, where $U_1, \ldots, U_r$ are non-isomorphic irreducible representations and each U_i has a_i multiplicities. Then if χ_i is the character of U_i, the character ψ of V is

$$\psi = a_1\chi_1 + \ldots + a_r\chi_r.$$

By property of inner products, this means that

1. $\langle \chi_i, \psi \rangle = a_i$
2. $\langle \psi, \psi \rangle = \sum_{i=1}^r a_i^2$.

Corollary

A representation V is irreducible if and only if $\langle \psi_V, \psi_V \rangle = 1$.

Let G be a group with l conjugacy classes. Once again, let $\mathbb{C}_{class}(G)$ be the set of all class functions. Then with the proper definition of addition and scalar multiplication $\mathbb{C}_{class}(G)$ is a vector space over $\mathbb{C}$. A basis for $\mathbb{C}_{class}(G)$ can be simply a set of functions that take the value 1 along a conjugacy class and zero elsewhere. Then $\dim \mathbb{C}_{class}(G) = l$.

Theorem

Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of a group G. Then $\chi_1, \ldots, \chi_k$ form a basis for $\mathbb{C}_{class}(G)$.

Corollary

The number of irreducible representations of a group is equal to the number of conjugacy classes for a group.

Theorem

Every representation is determined up to isomorphism by its character.

Proof.

Suppose two representations V and W are isomorphic. Then they share a decomposition such that $V \cong W \cong U_1^{\oplus a_1} \oplus \ldots \oplus U_r^{\oplus a_r}$, where the U_i's are distinct irreducible representations and the a_i's are their multiplicities. Thus the characters for V and W are both given by $a_1\chi_1 + \ldots + a_r\chi_r$. Conversely, suppose we are given that $\chi_V = \chi_W$. We can decompose V and W into $V \cong U_1^{\oplus a_1} \oplus \ldots \oplus U_r^{\oplus a_r}$ and $W \cong U_1^{\oplus b_1} \oplus \ldots \oplus U_r^{\oplus b_r}$. Since $\chi_V = \chi_W$, it follows that $a_1\chi_1 + \ldots + a_r\chi_r = b_1\chi_1 + \ldots + b_r\chi_r$. Now because the χ_i's are linearly independent, we observe that $a_i = b_i$ for every i. Hence $V \cong W$. □

Theorem (Column Orthogonality Relation)

Suppose $\chi_1, \ldots, \chi_k$ are all the irreducible characters of a group G. Let $g_1, \ldots, g_k$ be representatives of the conjugacy classes of G. Then for all g_i, g_j,

$$\sum_{l=1}^{k} \chi_l(g_i)\overline{\chi_l(g_j)} = \delta_{ij}\frac{|G|}{c(g_j)},$$

where $c(g_j)$ denotes the cardinality of the conjugacy class of g_j.

Proof.

For $1 \leq j \leq k$, define ψ_j to be the class function

$$\psi_j(g_i) = \delta_{ij}$$

where $1 \leq i \leq k$. By theorem, ψ_j is a linear combination of $\chi_1, \ldots, \chi_k$ so that

$$\psi_j = \sum_{l=1}^{k} \alpha_l \chi_l \quad (\alpha_l \in \mathbb{C}).$$

To obtain each α_l, we simply take the inner product $\langle \psi_j, \chi_l \rangle$ so that

$$\alpha_l = \langle \psi_j, \chi_l \rangle = \frac{1}{|G|} \sum_{g \in G} \psi_j(g) \overline{\chi_l(g)}.$$

$\psi_j(g) = 1$ if and only if g is conjugate to g_j and zero otherwise. $c(g_j)$ denotes the number of elements in G conjugate to g_j.

Proof.

Thus, we have

$$\alpha_l = \frac{1}{|G|}\sum_{g\in G}\psi_j(g)\overline{\chi_l(g)} = \frac{1}{|G|}c(g_j)\overline{\chi_l(g_j)},$$

from which we conclude

$$\delta_{ij} = \psi_j(g_i) = \sum_{l=1}^{k}\alpha_l\chi_l(g_i) = \frac{1}{|G|}\sum_{l=1}^{k}c(g_j)\chi_l(g_i)\overline{\chi_l(g_j)}.$$

Rearrange the equation and the result follows. □

Character tables

Character tables provide character values along each conjugacy class for each irreducible character.

Remark

All normal subgroups of G can be recognised from its character table; the kernel of a character is a normal subgroup and each normal subgroup is the intersection of the kernels of some of the irreducible characters of G.

We construct character tables for groups S_4 and $\mathbb{Z}_3$.

Character table for S_4

- There are five conjugacy classes for S_4, and we can give them representatives 1, (12), (123), (1234), $(12)(34)$.
- From the theorem, $|S_4| = 4! = 24 = \sum_{i=1}^{k} (\chi_i(1))^2$.
- We have the trivial representation ρ_1.
- We have the alternating representation $\rho_2(g)v = \mathrm{sgn}(g)v$ for $v \in V_2$, $\dim V_2 = 1$.
- We have the standard representation $\rho_3(g)v_i = v_{gi}$ for $v \in V_3 = \langle v_1, v_2, v_3, v_4 \rangle / \langle v_1 + v_2 + v_3 + v_4 \rangle$, $\dim V_3 = 3$.
- We have another irreducible representation obtained by taking the tensor product of the standard representation with the alternating representation.

Character table for S_4

g_i	1	(12)	(123)	(1234)	(12)(34)
$c(g_i)$	1	6	8	6	3
trivial χ_1	1	1	1	1	1
alternating χ_2	1	-1	1	-1	1
standard χ_3	3	1	0	-1	-1
$\chi_3 \otimes \chi_2$	3	-1	0	1	-1

To find the last character χ_5, we simply need to apply the column orthogonality relation to the columns, for example:

$$\begin{aligned} \sum_{l=1}^{k} \chi_l(1)\overline{\chi_l(1)} &= \frac{4!}{1} = 24 \\ 1^2 + 1^2 + 3^2 + 3^2 + (\chi_5(1))^2 &= 24 \\ (\chi_5(1))^2 &= 4 \\ \chi_5(1) &= 2 \end{aligned}$$

Character table for S_4

g_i	1	(12)	(123)	(1234)	(12)(34)
$c(g_i)$	1	6	8	6	3
trivial χ_1	1	1	1	1	1
alternating χ_2	1	-1	1	-1	1
standard χ_3	3	1	0	-1	-1
$\chi_3 \otimes \chi_2$	3	-1	0	1	-1
χ_5	2	0	-1	0	2

Character table for $\mathbb{Z}_3$

- $\mathbb{Z}_3$ has three conjugacy classes with one element each.
- We have the trivial representation ρ_1.
- We have the representation $\rho_2(g)v = [\omega^g]v$ for $v \in V$, $\dim V = 1$, where ω is a third root of unity.
- The third character can be obtained from the orthogonality relations.

Character table for $\mathbb{Z}_3$

g_i	0	1	2
$c(g_i)$	1	1	1
trivial χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

Thank you for attention.