

REPRESENTATION THEORY AND FLAT MANIFOLDS

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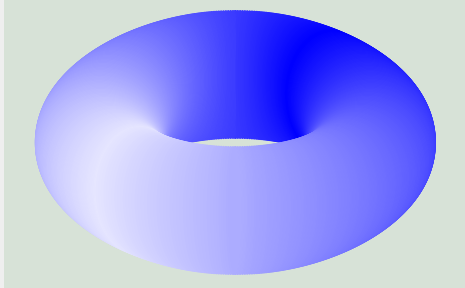
ANDRZEJ JANKOWSKI
MEMORIAL LECTURE
MINI CONFERENCE

MAY 10 - 12, 2019

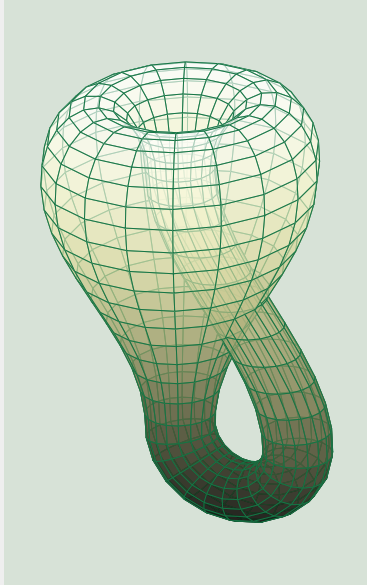
FLAT MANIFOLDS

TWO FLAT MANIFOLDS

Torus T

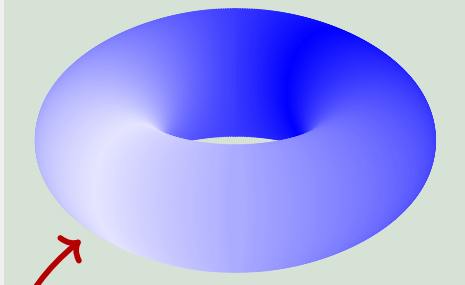


Klein bottle K

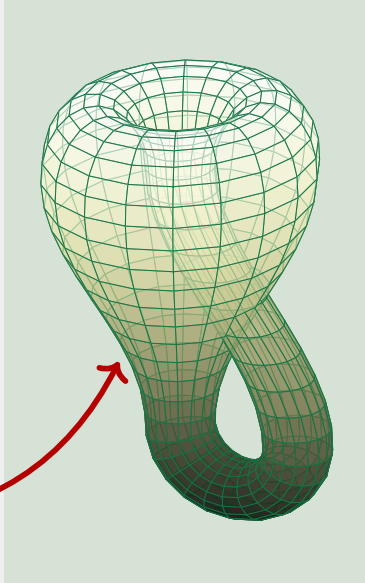


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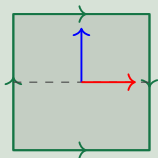
Klein bottle K



The only 2-dimensional
flat manifolds

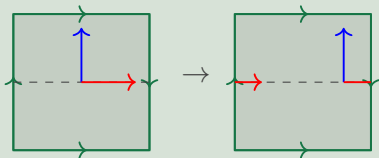
PARALLEL TRANSPORT

An element of a holonomy group of the Klein bottle



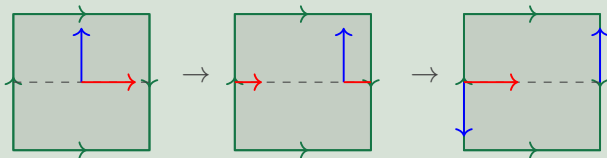
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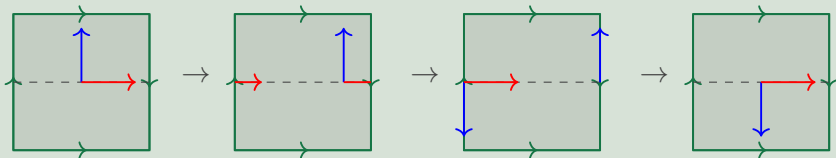
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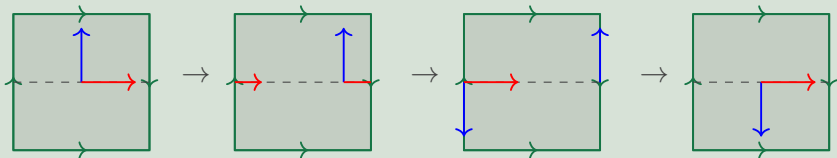
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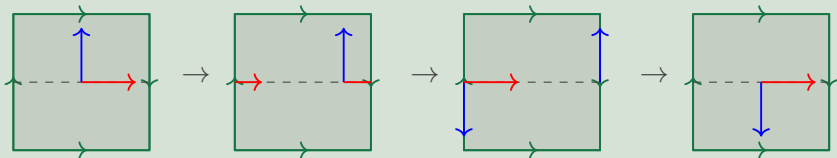


We get a matrix of the transformation:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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Holonomy group

Group of parallel transformations along all loops.

DECK TRANSFORMATIONS

$$A(n) = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$$

$$a + A : a \in \mathbb{R}^n, A \in \mathrm{GL}_n(\mathbb{R})$$

$$(a + A)(b + B)$$

$$a + Ab + B$$

$$(a + A)x$$

$$a + Ax$$

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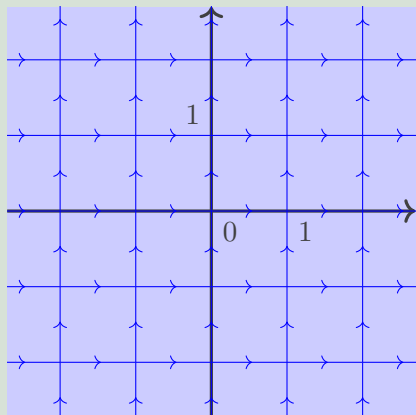
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$$\tilde{T} = \mathbb{R}^2$$



Deck transformations

$$\Gamma_T = \{z + \mathbb{I}_2 : z \in \mathbb{Z}^2\}$$

$$\Gamma_T \cong \pi_1(T) \cong \mathbb{Z}^2$$

$$T = \mathbb{R}^2 / \mathbb{Z}^2$$

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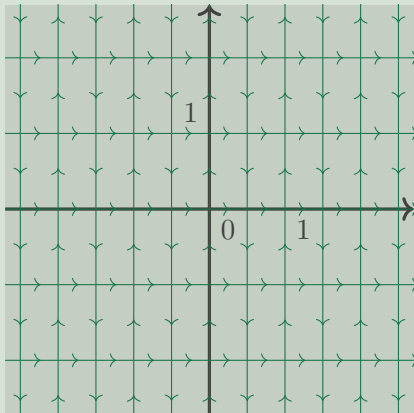
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$$\tilde{K} = \mathbb{R}^2$$



Deck transformations

$$[\Gamma_K : \Gamma_T] = 2$$

Missing element

$$\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Gamma_K \cong \pi_1(K)$$

$$K = \mathbb{R}^2 / \Gamma_K$$

CRYSTALLOGRAPHIC GROUPS

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Γ_T, Γ_K – examples of **crystallographic groups**

Discrete and cocompact subgroups of $E(n) := \mathbb{R}^n \rtimes O(n) \subset A(n)$.

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1. *Let $\Gamma \subset E(n)$ be an n -dimensional crystallographic group. The subgroup $\Gamma \cap (\mathbb{R}^n \times 1)$ of pure translations of Γ is free abelian group of rank n . Moreover it is maximal abelian normal subgroup of Γ of finite index.*

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2. *For every $n \in \mathbb{N}$ there is a finite number of isomorphism classes of crystallographic groups of dimension n .*
3. *Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group $A(n)$.*

STRUCTURE OF CRYSTALLOGRAPHIC GROUPS

$\Gamma \subset E(n)$ – crystallographic group

- Γ fits into a short exact sequence

$$0 \longrightarrow L \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- G – finite group – **holonomy group** of Γ .
- L – faithful G -lattice ($L \cong \mathbb{Z}^n$).
- We get an **integral holonomy representation** $\varphi: G \rightarrow \text{GL}(L)$:

$$\forall z \in L \subset \Gamma \forall g \in G \varphi_g(z) = \bar{g}z\bar{g}^{-1},$$

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Torus

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \Gamma_T \longrightarrow 1 \longrightarrow 1$$

Klein bottle

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \Gamma_K \longrightarrow C_2 \longrightarrow 1$$

BIEBERBACH GROUPS AND FLAT MANIFOLDS

Γ -torsionfree crystallographic

$X = \mathbb{R}^n / \Gamma$ - flat manifold

BIEBERBACH GROUPS AND FLAT MANIFOLDS

Γ - Bieberbach group

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$\text{Hol}(X)$ - holonomy gp of X

$$\text{Hol}(X)^{\text{GL}_n(\mathbb{R})} = \varphi(L)^{\text{GL}_n(\mathbb{R})}$$

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$\text{res}_H^G \alpha \neq 0$ for every $H < G$.

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$N := N_{\text{GL}(L)}(G) \curvearrowright H^2(G, L)$

$n \in N, a, b \in G, f$ - 2-cocycle:

$$n * f(a, b) = n \cdot f(a^n, b^n)$$

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$\mathcal{X} = \{\text{rep } H^G : H < G, |H| \text{ - prime}\}$

α - special:

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HOLONOMY GROUPS

CRYSTALLOGRAPHIC GROUPS REDEFINED

Theorem (Auslander, Kuranishi 1957)

Let Γ be a group which fits into a short exact sequence

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where G – finite, L – faithful G -lattice of \mathbb{Z} -rank n . Then there exists a monomorphism

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Idea of proof:

Every finite subgroup G of $GL_n(\mathbb{R})$ is orthogonal with respect to some inner product in \mathbb{R}^n , i.e.

$$\exists_{M \in GL_n(\mathbb{R})} G^M \subset O(n).$$

HOLONOMY GROUPS

Shapiro lemma: G – finite group, $H < G$, L – H -lattice

$$H^*(H, L) \cong H^*(G, \operatorname{ind}_H^G L)$$

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$H^2(G, L)$ contains a special element.

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Theorem (Auslander, Kuranishi 1957)

Every finite group is a holonomy group of some Bieberbach group (flat manifold).

SPECIAL ELEMENTS

HOLONOMY REPRESENTATION

Theorem (Hiss, Szczepański 1991)

Let $G \neq 1$ be a finite group. If L is an irreducible G -lattice then

$$H^2(G, L)$$

does not contain any special element.

Corollary

1. The only Bieberbach group with irreducible integral holonomy representation is \mathbb{Z} .
2. The only flat manifold with \mathbb{Q} -irreducible holonomy representation is T^1 .

SKETCH OF THE PROOF

p -special element $\alpha \in H^2(G, L)$

$\text{res}_H \alpha \neq 0$ for every $H \in \mathcal{X}_p = \{H \in \mathcal{X} : |H| = p\}$

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Lemma (Plesken 1989)

If indecomposable $\mathbb{Z}_p G$ -module M is not in the principal $\mathbb{Z}_p G$ -block then

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Assume L is irreducible and L_p contains an indecomposable direct summand in the principal $\mathbb{Z}_p G$ -block. Then every irreducible constituent of $\mathbb{C} \otimes L$ is in the principal p -block of G .

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Corollary: $\alpha \in H^2(G, L)$ – p -special

Some constituent of $\mathbb{C} \otimes L$ lies in the principal p -block of G .

EXAMPLE: $G = A_5$

Character table

	$1a$	$2a$	$3a$	$5a$	$5b$
χ_1	1	1	1	1	1
χ_2	3	-1	0	A	$*A$
χ_3	3	-1	0	$*A$	A
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$$A = (1 - \sqrt{5})/2, \quad *A = (1 + \sqrt{5})/2$$

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2-blocks

χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	2	1

3-blocks

χ_1	χ_2	χ_3	χ_4	χ_5
1	2	3	1	1

5-blocks

χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	1	2

SKETCH OF THE PROOF CONTINUED

M – G -lattice with character χ_M

$$\text{Irr}(G, M) = \{\chi \in \text{Irr } G : \langle \chi_M, \chi \rangle \neq 0\}$$

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$\mathcal{N} = \{N \triangleleft G : N\text{-minimal}\}$

$$\text{Soc } G = \langle \mathcal{N} \rangle = \prod_{N \in \mathcal{N}} N$$

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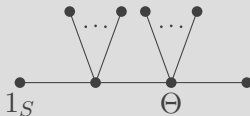
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Lemma (Hiss, Szczepański 1991)

L -irreducible, $\alpha \in H^2(G, L)$ – special, $S \triangleleft \text{Soc } G$ – simple:

- $\vartheta \in \text{Irr}(G, L)$: ϑ is in the principal p -block for every $p \mid \#G$
- $\psi \in \text{Irr}(S, L)$: ψ is in the principal p -block for every $p \mid \#S$
- $\text{Syl}_p(S)$ cyclic: there exists $\Theta \in \text{Irr}(S, L)$ with the following position on the Brauer tree:



EXAMPLE: $G = A_5, p = 5$

Three Brauer characters

$\varphi_1, \varphi_2, \varphi_3$

Restricted table

	$1a$	$2a$	$3a$
χ'_1	1	1	1
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Combinations

$$\begin{aligned}\chi'_1 &= \varphi_1 \\ \chi'_2 &= \varphi_2 \\ \chi'_3 &= \varphi_2 \\ \chi'_4 &= \varphi_1 + \varphi_2 \\ \chi'_5 &= \varphi_3\end{aligned}$$

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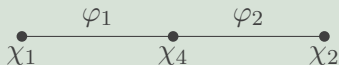
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Brauer tree (of the principal block)



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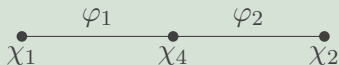
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$\chi'_2 = \chi'_3$: χ_2, χ_3 label the same node

EXAMPLE: $G = A_5$, SPECIAL ELEMENTS

$L - G$ -lattice with character χ_L

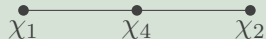
$\alpha \in H^2(G, L) - \text{special}$

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χ_5	5	1	-1	0	0

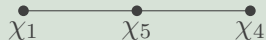
$$A = (1 - \sqrt{5})/2, \quad *A = (1 + \sqrt{5})/2$$

Brauer tree, $p = 5$



$$\langle \chi_L, \chi_2 \rangle, \langle \chi_L, \chi_3 \rangle > 0$$

Brauer tree, $p = 3$

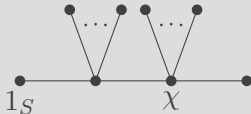


$$\langle \chi_L, \chi_4 \rangle > 0$$

SKETCH OF THE PROOF: (ALMOST) THE END

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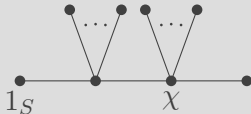
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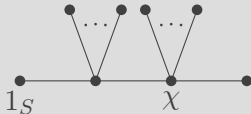
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Proof uses CFSG.

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Homogeneous G -lattice L :

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Corollary

1. The only Bieberbach groups with homogeneous integral holonomy representation are f.g. free abelian groups.
2. The only flat manifolds with \mathbb{Q} -homogeneous holonomy representation are flat tori.

SOME QUESTIONS

Theorem

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Question (Szczepański 2006)

Can we construct a faithful G -lattice L with a special element s.t. $\mathbb{Q} \otimes L$ is multiplicity-free?

SYMMETRIES OF FLAT MANIFOLDS

SYMMETRIES

Recall: $X = \mathbb{R}^n / \Gamma$

$$0 \longrightarrow L \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

defined by a special element $\alpha \in H^2(G, L)$

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Affine self-equivalences of X

$$\text{Aff}(X) := \{f: X \rightarrow X : \tilde{f} \in A(n)\}$$

DIAGRAMS

Theorem (Charlap, Vasquez 1973)

$$\begin{array}{ccccccc} & & 0 & & 1 & & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L^G & \longrightarrow & \Gamma & \longrightarrow & \text{Inn}(\Gamma) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\mathbb{R}^n)^G & \longrightarrow & N(\Gamma) & \longrightarrow & \text{Aut}(\Gamma) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \text{Aff}_0(X) & \longrightarrow & \text{Aff}(X) & \longrightarrow & \text{Out}(\Gamma) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 1 & & 1 \end{array}$$

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Remark

- $L^G, (\mathbb{R}^n)^G$ – fixed points of the G -action
- $N(\Gamma)$ – normalizer of Γ in $A(n)$
- $\beta_1 = \text{rank } L^G = \dim (\mathbb{R}^n)^G: \text{Aff}_0(X) \cong (S^1)^{\beta_1}$

DIAGRAMS

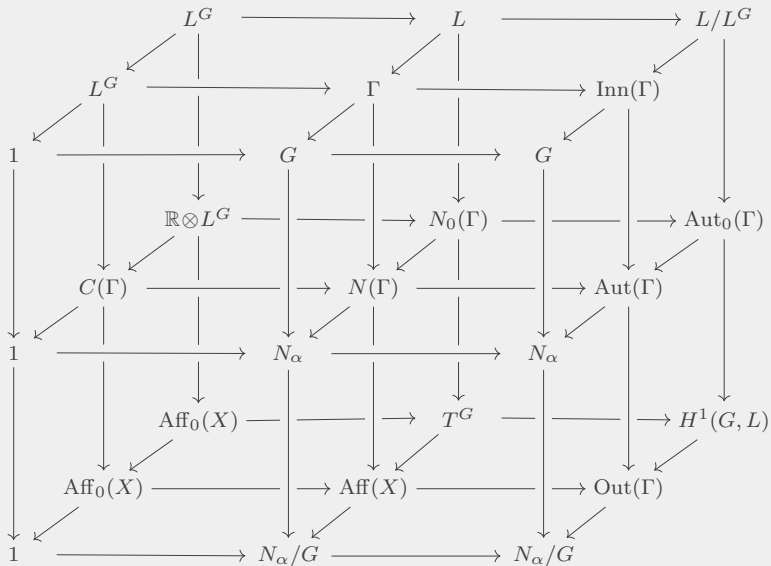
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Remark

- Automorphisms from $\text{Aut}^0(\Gamma)$ induce identities on L and G .
- N_α – stabilizer of α in $N_{\text{GL}(L)}(G)$

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Theorem (Szczepański 1993)

$\text{Out}(\Gamma)$ is finite iff every simple component of $\mathbb{Q} \otimes L$:

- is of multiplicity one
- is \mathbb{R} -irreducible

EXAMPLE: $G = A_5$ AGAIN

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$\mathbb{R} \otimes V$ splits:

- If Γ is a Bieberbach group with holonomy gp A_5 then $\text{Out}(\Gamma)$ is infinite.
- If X is a flat manifold with holonomy gp A_5 then $\text{Aff}(X)$ is infinite.

SOME \mathfrak{R}_1 -GROUPS

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Theorem (Hiss, Szczepański 1994)

Non \mathfrak{R}_1 -groups

- A_5
- *some 2 and 3-groups*
- *abelian G for $\exp(G) \nmid 12$*
- *D_{2n} for $n \nmid 12$*
- $\text{PSL}(2, p), p \notin \{2, 3, 7\}$
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Question (Szczepański 2006)

Is every finite group an outer automorphism group of a Bieberbach group with trivial center?

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For $n > 2$

$\text{Spin}(n)$ – universal cover of $\text{SO}(n)$

EXAMPLE: Spin(4)

$$\lambda(e_1e_2) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\lambda(e_2e_4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\lambda\left(\frac{1+e_1e_2}{\sqrt{2}}\right) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Pfäffle 2000: algebraic criterion

Spin structure on Γ is a homomorphism $\varepsilon: \Gamma \rightarrow \text{Spin}(n)$:

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If $\pi' : \Gamma \rightarrow \text{SO}(n)$ is equivalent (in $\text{GL}_n(\mathbb{R})$) to π , then

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Lemma (Dekimpe, Sadowski, Szczepański 2006)

Γ has a spin structure iff $\pi^{-1}(\text{Syl}_2(G))$ has one.

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Theorem (Eckmann, Mislin 1979)

Let G be a finite p -group. Then every \mathbb{Q} -irreducible representation of G is either induced from a representation of a subgroup of index p or it factors through a representation of a cyclic group of order p .

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Every rational representation $\tau: G \rightarrow \mathrm{GL}_n(\mathbb{Q})$ of 2-group G is equivalent to a representation $\rho: G \rightarrow \mathrm{O}_n(\mathbb{Z}) = \mathrm{O}(n) \cap \mathrm{GL}_n(\mathbb{Z})$.

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Thank you!