Representation theory and flat manifolds

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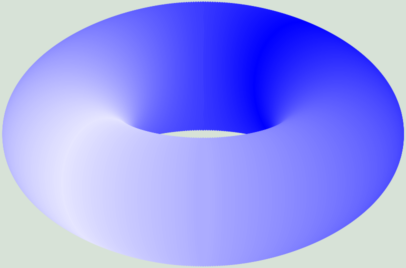
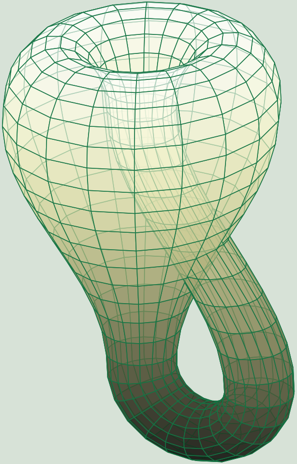
Flat manifolds

Two flat manifolds

Torus T



Klein bottle K



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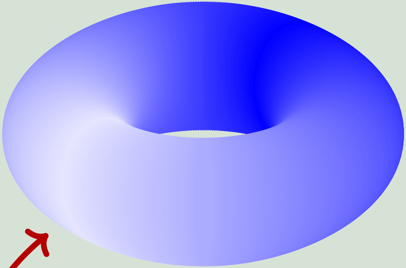

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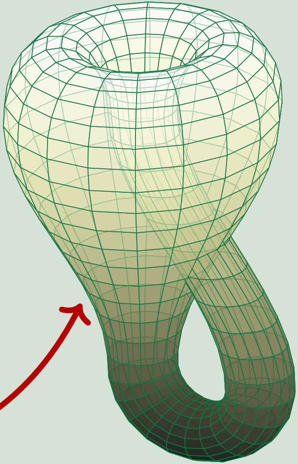
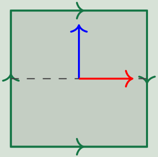


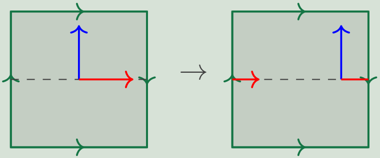
The only 2-dimensional flat manifolds

An element of a holonomy group of the Klein bottle



Parallel transport

An element of a holonomy group of the Klein bottle



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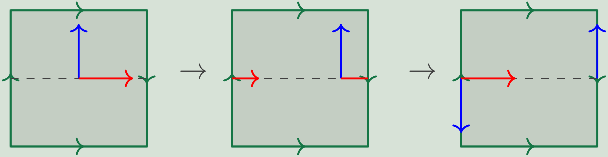

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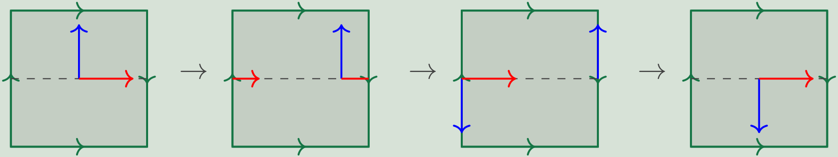

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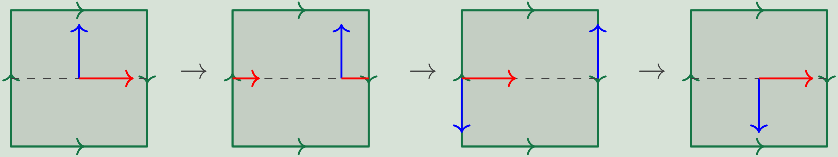

We get a matrix of the transformation:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Parallel transport

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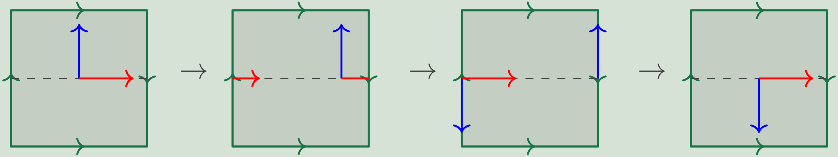

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Holonomy group

Group of parallel transformations along all loops.

Deck transformations

$\mathrm{A}(n) = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$	$(a+A)(b+B)$	$(a+A)x$
$a + A : a \in \mathbb{R}^n, A \in \mathrm{GL}_n(\mathbb{R})$	$a + Ab + B$	$a + Ax$

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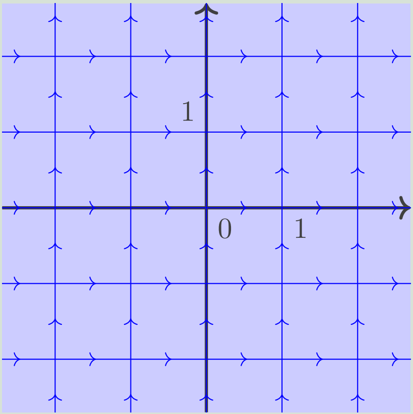
$\widetilde{T} = \mathbb{R}^2$



Deck transformations

$$\Gamma_T = \{z + \mathbb{I}_2 : z \in \mathbb{Z}^2\}$$

$\Gamma_T \cong \pi_1(T) \cong \mathbb{Z}^2$

$$T = \mathbb{R}^2/\mathbb{Z}^2$$

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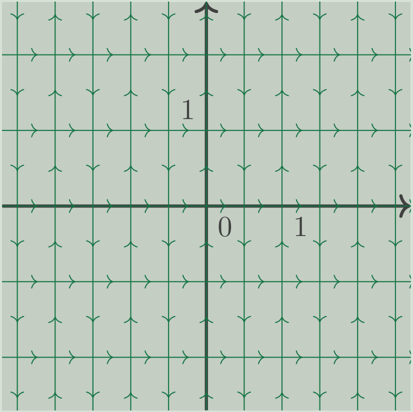

Deck transformations

$$[\Gamma_K : \Gamma_T] = 2$$

Missing element

$$\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\Gamma_K \cong \pi_1(K)$

$$K = \mathbb{R}^2/\Gamma_K$$

Crystallographic groups

Crystallographic groups

Γ_T, Γ_K – examples of crystallographic groups

Discrete and cocompact subgroups of $\mathrm{E}(n) := \mathbb{R}^n \rtimes \mathrm{O}(n) \subset \mathrm{A}(n)$.

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2. *For every $n \in \mathbb{N}$ there is a finite number of isomorphism classes of crystallographic groups of dimension n.*
3. *Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group $\mathrm{A}(n)$.*

Structure of crystallographic groups

$\Gamma \subset E(n)$ – crystallographic group

- Γ fits into a short exact sequence
$$0 \longrightarrow L \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$
- G – finite group – holonomy group of Γ.
- L – faithful G-lattice ($L \cong \mathbb{Z}^n$).
- We get an integral holonomy representation $\varphi\colon G \to \mathrm{GL}(L)$:
$$\forall_{z \in L \subset \Gamma} \forall_{g \in G}\ \varphi_g(z) = \overline{g} z \overline{g}^{-1},$$
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Torus

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \Gamma_T \longrightarrow 1 \longrightarrow 1$$

Klein bottle

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \Gamma_K \longrightarrow C_2 \longrightarrow 1$$

Bieberbach groups and flat manifolds

Γ–torsionfree crystallographic

$X = \mathbb{R}^n/\Gamma$ – flat manifold

Bieberbach groups and flat manifolds

Γ–Bieberbach group

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$\mathrm{Hol}(X)$ – holonomy gp of X

$$\mathrm{Hol}(X)^{\mathrm{GL}_n(\mathbb{R})} = \varphi(L)^{\mathrm{GL}_n(\mathbb{R})}$$

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$\mathrm{res}^G_H \alpha \neq 0$ for every $H < G$.

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$$n * f(a, b) = n \cdot f(a^n, b^n)$$

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$\mathcal{X} = \{\mathrm{rep}\, H^G : H < G, |H| - \text{prime}\}$

α – special:

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Holonomy groups

Crystallographic groups redefined

Theorem (Auslander, Kuranishi 1957)

Let Γ be a group which fits into a short exact sequence

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where G – finite, L – faithful G-lattice of $\mathbb{Z}$-rank n. Then there exists a monomorphism

$$F\colon \Gamma \to E(n)$$

st. $F(\Gamma) \subset E(n)$ is crystallographic.

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st. $F(\Gamma) \subset E(n)$ *is crystallographic.*

Idea of proof:

Every finite subgroup G of $\mathrm{GL}_n(\mathbb{R})$ is orthogonal with respect to some inner product in $\mathbb{R}^n$, i.e.

$$\exists_{M \in \mathrm{GL}_n(\mathbb{R})} G^M \subset \mathrm{O}(n).$$

Holonomy groups

Shapiro lemma: G – finite group, $H < G$, L – H-lattice

$$H^*(H, L) \cong H^*(G, \operatorname{ind}_H^G L)$$

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Note

L may be non-faithful.

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Theorem (Auslander, Kuranishi 1957)

Every finite group is a holonomy group of some Bieberbach group (flat manifold).

Special elements

Holonomy representation

Theorem (Hiss, Szczepański 1991)

Let $G \neq 1$ be a finite group. If L is an irreducible G-lattice then

$$H^2(G, L)$$

does not contain any special element.

Corollary

1. *The only Bieberbach group with irreducible integral holonomy representation is $\mathbb{Z}$.*
2. *The only flat manifold with $\mathbb{Q}$-irreducible holonomy representation is T^1.*

Sketch of the proof

p-special element $\alpha \in H^2(G, L)$

$\operatorname{res}_H \alpha \neq 0$ for every $H \in \mathcal{X}_p = \{H \in \mathcal{X} : |H| = p\}$

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Lemma (Plesken 1989)

If indecomposable $\mathbb{Z}_p G$-module M is not in the principal $\mathbb{Z}_p G$-block then

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Assume L is irreducible and L_p contains an indecomposable direct summand in the principal $\mathbb{Z}_pG$-block. Then every irreducible constituent of $\mathbb{C} \otimes L$ is in the principal p-block of G.

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Corollary: $\alpha \in H^2(G, L)$ – p-special

Some constituent of $\mathbb{C} \otimes L$ lies in the principal p-block of G.

EXAMPLE: $G = A_5$

Character table

	$1a$	$2a$	$3a$	$5a$	$5b$
χ_1	1	1	1	1	1
χ_2	3	-1	0	A	$^*A$
χ_3	3	-1	0	$^*A$	A
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$A = (1-\sqrt{5})/2,\ {}^*A = (1+\sqrt{5})/2$

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2-blocks

χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	2	1

3-blocks

χ_1	χ_2	χ_3	χ_4	χ_5
1	2	3	1	1

5-blocks

χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	1	2

Sketch of the proof continued

M – G-lattice with character χ_M

$$\mathrm{Irr}(G,M)=\{\chi\in\mathrm{Irr}\,G : \langle\chi_M,\chi\rangle\neq 0\}$$

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$$\operatorname{Soc}G=\langle\mathcal{N}\rangle=\prod_{N\in\mathcal{N}}N$$

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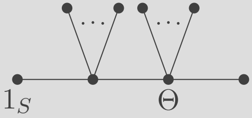
Lemma (Hiss, Szczepański 1991)

L-irreducible,$\alpha\in H^2(G,L)$ – special, $S\lhd\mathrm{Soc}\,G$ – simple:

- $\vartheta\in\mathrm{Irr}(G,L)$: ϑ is in the principal p-block for every $p|\#G$
- $\psi\in\mathrm{Irr}(S,L)$: ψ is in the principal p-block for every $p|\#S$
- $\mathrm{Syl}_p(S)$ cyclic: there exists $\Theta\in\mathrm{Irr}(S,L)$ with the following position on the Brauer tree:



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Three Brauer characters

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Combinations

$$\begin{aligned}
\chi_1' &= \varphi_1 \\
\chi_2' &= \varphi_2 \\
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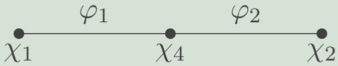
Brauer tree (of the principal block)



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Brauer tree (of the principal block)

$$\begin{tikzcd}
\chi_1 \arrow[r, dash, "\varphi_1"] & \chi_4 \arrow[r, dash, "\varphi_2"] & \chi_2
\end{tikzcd}$$

$\chi_2' = \chi_3'$: χ_2, χ_3 label the same node

Example: $G = A_5$, special elements

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Character table

	$1a$	$2a$	$3a$	$5a$	$5b$
χ_1	1	1	1	1	1
χ_2	3	-1	0	A	$*A$
χ_3	3	-1	0	$*A$	A
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$A = (1-\sqrt{5})/2,\ {}^*A = (1+\sqrt{5})/2$

Brauer tree, $p = 5$

$$\begin{tikzcd} \chi_1 \arrow[r, dash] & \chi_4 \arrow[r, dash] & \chi_2 \end{tikzcd}$$

$\langle \chi_L, \chi_2 \rangle, \langle \chi_L, \chi_3 \rangle > 0$

Brauer tree, $p = 3$

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Sketch of the proof: (almost) the end

$\mathcal{S}(G) \subset \operatorname{Irr} G$. $\chi \in \mathcal{S}(G)$ if for every $p|\#G$:

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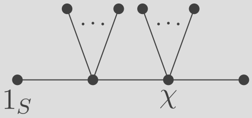
- If $\operatorname{Syl}_p(G)$ is cyclic: χ has the following position on the Brauer tree:



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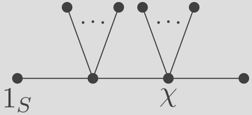

Proposition (Hiss, Szczepański 1991)

If S is a finite nonabelian simple group, then $\mathcal{S}(S) = \emptyset$.

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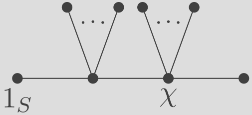

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Remark

Proof uses CFSG.

Homogeneous representations

Homogeneous G-lattice L:

$$\mathbb{Q} \otimes L = V \oplus \ldots \oplus V$$

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Let $G \neq 1$ be a finite group. If L is a homogeneous G-lattice then

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Corollary

1. *The only Bieberbach groups with homogeneous integral holonomy representation are f.g. free abelian groups.*
2. *The only flat manifolds with $\mathbb{Q}$-homogeneous holonomy representation are flat tori.*

Some questions

Theorem

Every finite group is a holonomy group of some flat manifold.

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What is the minimum number of $\mathbb{Q}$-homogeneous components of a faithful G-lattice L for which we can find a special element?

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What is the minimum number of $\mathbb{Q}$-homogeneous components of a faithful G-lattice L for which we can find a special element?

Question (Szczepański 2006)

Can we construct a faithful G-lattice L with a special element s.t. $\mathbb{Q} \otimes L$ is multiplicity-free?

Symmetries of flat manifolds

Symmetries

Recall: $X = \mathbb{R}^n/\Gamma$

$$0 \longrightarrow L \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

defined by a special element $\alpha \in H^2(G, L)$

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Affine self-equivalences of X

$$\mathrm{Aff}(X) := \{f \colon X \to X : \tilde{f} \in \mathrm{A}(n)\}$$

Diagrams

Theorem (Charlap, Vasquez 1973)

$$\begin{tikzcd}
 & 0 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
0 \arrow[r] & L^G \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \mathrm{Inn}(\Gamma) \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & (\mathbb{R}^n)^G \arrow[r] \arrow[d] & N(\Gamma) \arrow[r] \arrow[d] & \mathrm{Aut}(\Gamma) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Aff}_0(X) \arrow[r] \arrow[d] & \mathrm{Aff}(X) \arrow[r] \arrow[d] & \mathrm{Out}(\Gamma) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

Diagrams

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\begin{array}{ccccc}
L^G & \longrightarrow & \Gamma & \longrightarrow & \mathrm{Inn}(\Gamma) \\
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\end{array}
$$

Remark

- $L^G, (\mathbb{R}^n)^G$ – fixed points of the G-action
- $N(\Gamma)$ – normalizer of Γ in $A(n)$
- $\beta_1 = \mathrm{rank}\, L^G = \dim (\mathbb{R}^n)^G$: $\mathrm{Aff}_0(X) \cong (S^1)^{\beta_1}$

Diagrams

Theorem (Charlap, Vasquez 1973)

$$\begin{tikzcd}
L/L^G \arrow[r] \arrow[d] & \mathrm{Inn}(\Gamma) \arrow[r] \arrow[d] & G \arrow[d] \\
\mathrm{Aut}^0(\Gamma) \arrow[r] \arrow[d] & \mathrm{Aut}(\Gamma) \arrow[r] \arrow[d] & N_\alpha \arrow[d] \\
H^1(G,L) \arrow[r] & \mathrm{Out}(\Gamma) \arrow[r] & N_\alpha/G
\end{tikzcd}$$

Remark

- Automorphisms from $\mathrm{Aut}^0(\Gamma)$ induce identities on L and G.
- N_α – stabilizer of α in $N_{\mathrm{GL}(L)}(G)$

Diagrams

$$
\begin{tikzcd}
 & & L^G \arrow[rrr] \arrow[dl] \arrow[ddd] & & & L \arrow[rrr] \arrow[dl] \arrow[ddd] & & & L/L^G \arrow[dl] \arrow[ddd] \\
 & L^G \arrow[rrr] \arrow[dl] \arrow[ddd] & & & \Gamma \arrow[rrr] \arrow[dl] \arrow[ddd] & & & \mathrm{Inn}(\Gamma) \arrow[dl] \arrow[ddd] & \\
1 \arrow[rrr] \arrow[ddd] & & & G \arrow[rrr] \arrow[ddd] & & & G \arrow[ddd] & & \\
 & & \mathbb{R}\otimes L^G \arrow[rrr] \arrow[dl] \arrow[ddd] & & & N_0(\Gamma) \arrow[rrr] \arrow[dl] \arrow[ddd] & & & \mathrm{Aut}_0(\Gamma) \arrow[dl] \arrow[ddd] \\
 & C(\Gamma) \arrow[rrr] \arrow[dl] \arrow[ddd] & & & N(\Gamma) \arrow[rrr] \arrow[dl] \arrow[ddd] & & & \mathrm{Aut}(\Gamma) \arrow[dl] \arrow[ddd] & \\
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Finiteness of $\mathrm{Out}(\Gamma)$

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Theorem (Szczepański 1993)

$\mathrm{Out}(\Gamma)$ *is finite iff every simple component of* $\mathbb{Q} \otimes L$*:*

- *is of multiplicity one*
- *is* $\mathbb{R}$*-irreducible*

EXAMPLE: $G = A_5$ AGAIN

L – G-lattice with character χ_L

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$\mathbb{R} \otimes V$ splits:

- If Γ is a Bieberbach group with holonomy gp A_5 then $\mathrm{Out}(\Gamma)$ is infinite.
- If X is a flat manifold with holonomy gp A_5 then $\mathrm{Aff}(X)$ is infinite.

Some $\mathfrak{R}_1$-groups

G – $\mathfrak{R}_1$-group

There is a Bieberbach gp Γ with holonomy G and finite $\mathrm{Out}(\Gamma)$.

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Non $\mathfrak{R}_1$-groups

- A_5
- *some* 2 *and* 3*-groups*
- *abelian* G *for* $\exp(G) \nmid 12$
- D_{2n} *for* $n \nmid 12$
- $\mathrm{PSL}(2,p), p \notin \{2,3,7\}$
- p*-groups,* $p \geq 5$

$\mathfrak{R}_1$-groups

- M_{11}
- *some* 2 *and* 3*-groups*
- *abelian* G *for* $\exp(G) \mid 12$
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Finite groups of symmetries

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Question (Szczepański 2006)

Is every finite group an outer automorphism group of a Bieberbach group with trivial center?

Examples: non exhaustive list

Zimmermann 1990

$$n = 3, G = C_2^2, |\mathrm{Out}(\Gamma)| = 96$$

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Calculation outer automorphisms up to dimension 6.

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Elementary abelian 2-groups.

Spin structures

Spin group

Assumption: $n \in \mathbb{N}$

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Real associative algebra with one, generated by $e_1, \ldots, e_n$ st.

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For $n > 2$

$\mathrm{Spin}(n)$ – universal cover of $\mathrm{SO}(n)$

Example: $\mathrm{Spin}(4)$

$$\lambda(e_1e_2) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\lambda(e_2e_4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\lambda\left(\frac{1+e_1e_2}{\sqrt{2}}\right) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Assume $\pi \colon \Gamma \to \mathrm{SO}(n)$

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Pfäffle 2000: algebraic criterion

Spin structure on Γ is a homomorphism $\varepsilon\colon \Gamma \to \mathrm{Spin}(n)$:

$$\begin{tikzcd} & \mathrm{Spin}(n) \arrow[d, "\lambda"] \\ \Gamma \arrow[ur, "\varepsilon"] \arrow[r, "\pi"'] & \mathrm{SO}(n) \end{tikzcd}$$

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Problem?

Determine $\lambda^{-1}(\pi(\Gamma))$.

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Theorem (Hiss, Szczepański 2008)

If $\pi' : \Gamma \to \mathrm{SO}(n)$ is equivalent (in $\mathrm{GL}_n(\mathbb{R})$) to π, then

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Lemma (Dekimpe, Sadowski, Szczepański 2006)

Γ *has a spin structure iff* $\pi^{-1}(\mathrm{Syl}_2(G))$ *has one.*

2-GROUPS

Theorem (Eckmann, Mislin 1979)

Let G be a finite p-group. Then every $\mathbb{Q}$-irreducible representation of G is either induced from a representation of a subgroup of index p or it factors through a representation of a cyclic group of order p.

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Why is $\mathrm{SO}_n(\mathbb{Z})$ good?

$$\mathrm{SO}_n(\mathbb{Z}) = \left\langle \lambda(e_i e_j), \lambda\left(\frac{1+e_i e_j}{\sqrt{2}}\right) : 1 \le i < j \le n \right\rangle$$

Thank you!