Spin structures on flat manifolds

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Bieberbach groups and flat manifolds

- $E(n) = O(n) \ltimes \mathbb{R}^n$ – isometry group of the Euclidean space $\mathbb{R}^n$.
- Discrete and cocompact subgroup $\Gamma \subset E(n)$ – crystallographic group.
- Torsionfree crystallographic $\Gamma \subset E(n)$ – Bieberbach group.
  - $X = \mathbb{R}^n/\Gamma$ – flat manifold (closed connected Riemannian n-manifold with zero sectional curvature).
  - $\pi_1(X) \cong \Gamma$.

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Structure of Bieberbach groups

Theorem (Bieberbach 1911)

The subgroup $\Gamma \cap (1 \times \mathbb{R}^n)$ of pure translations of Γ is free abelian group of rank n. Moreover it is maximal abelian normal subgroup of Γ of finite index.

Structure of Bieberbach groups

- Γ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- G – finite group – holonomy group of Γ (X).
- $\pi\colon \Gamma \to G \subset SO(n)$:

$$\forall_{(A,a)\in\Gamma}\ \pi(A,a) = A.$$

- We get a holonomy representation $\varphi\colon G \to \mathrm{GL}(n,\mathbb{Z})$:

$$\forall_{z\in\mathbb{Z}^n\subset\Gamma}\forall_{g\in G}\ \varphi_g(z) = \overline{g}z\overline{g}^{-1},$$

where $\pi(\overline{g}) = g$.
- φ is $\mathbb{R}$-equivalent to $id\colon G \to G \subset \mathrm{SO}(n)$.

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Clifford algebra

Definition

Let $n \in \mathbb{N}$. The Clifford algebra C_n is a real associative algebra with one, generated by elements $e_1, \ldots, e_n$, which satisfy relations:

$$\forall_{1 \leq i < j \leq n} \; e_i^2 = -1 \wedge e_i e_j = -e_j e_i.$$

- $C_0 = \mathbb{R}, C_1 = \mathbb{C}, C_2 = \mathbb{H}$.
- $\mathbb{R}^n = \operatorname{span}\{e_1, \ldots, e_n\} \subset C_n$.

Definition (Three involutions)

- $(e_{i_1} \ldots e_{i_k})^* = e_{i_k} \ldots e_{i_1}$.
- $e_i' = -e_i$.
- $\overline{a} = (a')^*, a \in C_n$.

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Spin group

$$\forall_{n\in\mathbb{N}}\ \mathrm{Spin}(n) := \{x \in C_n \mid x' = x \wedge x\overline{x} = 1\}.$$

Proposition

Let $n \in \mathbb{N}$. The map $\lambda_n \colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$, defined by

$$\forall_{x\in\mathrm{Spin}(n)}\forall_{v\in\mathbb{R}^n}\ \lambda_n(x)v = xv\overline{x}$$

is a continuous group epimorphism.

For $n \geq 3$:

- $\mathrm{Spin}(n)$ – universal cover of $\mathrm{SO}(n)$.
- $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2\mathbb{Z}$.
- $\ker \lambda_n = \{\pm 1\}$.

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Spin structures on Riemannian manifolds

Definition

Let M be an n-dimensional oriented Riemannian manifold. A spin structure on M is a λ_n extension of the oriented orthonormal frame bundle of M ($\mathrm{SO}(n)$ principal bundle) to the $\mathrm{Spin}(n)$ bundle.

Example

Every compact oriented 3 manifold admits a spin structure (it has trivial tangent bundle).

Proposition

M admits a spin structure if and only if $w_2(M) = 0$.

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Algebraic condition

Proposition (Pfaffle 1999)

Let $\Gamma \in E(n)$ *be a Bieberbach group. Then the set of spin structures on the flat manifold* $\mathbb{R}^n/\Gamma$ *is in bijection with the set of the homomorphisms of the form* $\epsilon\colon \Gamma \to \mathrm{Spin}(n)$ *which satisfy* $\lambda_n \epsilon = \pi$*:*

$$
\begin{array}{ccccccccc}
 & & & & & & \mathrm{Spin}(n) & & \\
 & & & & & \overset{\epsilon}{\nearrow} & \downarrow \lambda_n & & \\
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\end{array}
$$

Determining spin structures

$$\begin{array}{ccc} & & \mathrm{Spin}(n) \\ & \overset{\epsilon}{\nearrow} & \big\downarrow \lambda_n \\ \Gamma & \xrightarrow{\pi} & G \end{array}$$

- Every crystallographic group is finitely presented. Let

$$\Gamma = \langle X \mid R \rangle,$$

  be a presentation of Γ, with finite X and R.

Determining spin structures

For all maps $\epsilon\colon X \to \lambda_n^{-1}\pi(X)$ for which $\lambda_n \epsilon = \pi$ check which preserve relations of Γ:

$$\forall_{r_1,\dots,r_l \in X \cup X^{-1}}\ r_1 \dots r_l \in R \overset{?}{\Rightarrow} \epsilon(r_1)\dots\epsilon(r_l) = 1.$$

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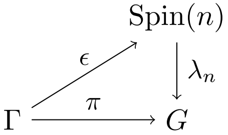

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A question

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How to determine $\lambda_n^{-1}\pi(X) \subset \lambda_n^{-1}(G)$?

- $G = \pi(\Gamma) \subset \mathrm{SO}(n)$ – finite group.
- For $n \geq 3$ $\ker \lambda_n = \{\pm 1\}$:

$$\forall_{x \in \mathrm{Spin}(n)} \forall_{g \in \mathrm{SO}(n)}\ \lambda_n(x) = g \Rightarrow \lambda_n^{-1}(g) = \{\pm x\}.$$

Remark

From now on we assume $n \geq 3$.

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Proposition (Hiss, Szczepański 2008)

Let $\Gamma_1, \Gamma_2 \subset E(n)$ *be isomorphic Bieberbach groups. Then the set of spin structures of* $\mathbb{R}^n/\Gamma_1$ *is in bijection with the set of spin structures of* $\mathbb{R}^n/\Gamma_2$*.*

Corollary

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$$\begin{array}{ccc} \Gamma & \xrightarrow{\epsilon} & \mathrm{Spin}(n) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \lambda_n} \\ G & \xrightarrow{\varphi} & \varphi(G) \end{array}$$

where $\varphi\colon G \to \mathrm{SO}(n)$ *is a representation of* G *which is* $\mathbb{R}$*-equivalent to* $id\colon G \to G \subset \mathrm{SO}(n)$*.*

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Existence of spin structures on flat manifolds

Necessary and sufficient condition

Lemma

Let Γ be an n-dimensional Bieberbach group with holonomy group G:

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

Let $F \subset G$ be a Sylow 2-subgroup of G. Then $\mathbb{R}^n/\Gamma$ has a spin structure if and only if $\mathbb{R}^n/\pi^{-1}(F)$ has one.

Corollary

It is enough to find "good" representation $\varphi\colon G \to \mathrm{SO}(n)$ with assumption that G is a 2-group.

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Theorem (Putrycz, Szczepański 2008)

24 out of the 27 oriented flat 4-manifolds have a spin structure.

Group of "good" matrices

$$\mathrm{O}(n,\mathbb{Z}) := \mathrm{GL}(n,\mathbb{Z}) \cap O(n), \quad \mathrm{SO}(n,\mathbb{Z}) := \mathrm{GL}(n,\mathbb{Z}) \cap \mathrm{SO}(n)$$

- $\mathcal{D} \subset \mathrm{GL}(n,\mathbb{Z})$ – subgroup of diagonal matrices (± 1 on diagonal).
- $P_\sigma \in \mathrm{GL}(n,\mathbb{Z})$ – matrix of a permutation $\sigma \in S_n$.

Lemma

$$\forall_{A\in(n,\mathbb{Z})} \exists_{D\in\mathcal{D}} \exists_{\sigma\in S_n} \; A = DP_\sigma$$

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- E_{ij} – matrix of the transposition $(i\ j)$ with -1 instead of 1 in the ith row, where $1 \leq i < j \leq n$.

Corollary

Let $A \in \mathrm{SO}(n,\mathbb{Z})$. *Then*

$$A = DE_{i_1j_1}\dots E_{i_kj_k},$$

where $D \in \mathcal{D} \cap \mathrm{SO}(n,\mathbb{Z})$, $i_l < j_l$ *for* $l = 1,\dots,k$.

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Preimages of "good" matrices

Lemma

Let $D \in \mathcal{D} \cap \mathrm{SO}(n, \mathbb{Z})$ *has* -1 *in the entries* $i_1 < \ldots < i_m$ *of the diagonal. Then*

$$\lambda_n(e_{i_1} \ldots e_{i_m}) = D.$$

Lemma

$$\forall_{1 \leq i < j \leq n} \; \lambda_n \left(\frac{1 + e_i e_j}{\sqrt{2}} \right) = E_{ij}.$$

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Every rational representation of 2-group is "good"

Theorem (Eckmann, Mislin 1979)

Let G be a finite p-group. Then every $\mathbb{Q}$-irreducible representation of G is either induced from a representation of a subgroup of index p or it factors through a representation of a cyclic group of order p.

Corollary A

Every rational representation $\rho\colon G \to \mathrm{GL}(k, \mathbb{Q})$ of 2-group G is equivalent to a representation $\varphi\colon G \to \mathrm{O}(k, \mathbb{Z})$.

Corollary B

Every rational representation $\rho\colon G \to \mathrm{SL}(k, \mathbb{Q})$ of 2-group G is equivalent to a representation $\varphi\colon G \to \mathrm{SO}(k, \mathbb{Z})$.

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Proof of Corollary A.

We may assume that ρ is irreducible.

1. The group $C_2 = \langle c \mid c^2 = 1 \rangle$ has exactly two irreducible representations:

$$c \mapsto 1, \quad c \mapsto -1.$$

□

Every rational representation of 2-group is "good"

Proof of Corollary A.

We may assume that ρ is irreducible.

2. Assume that the statement is true for every 2-group of order less than $|G|$. Let $\rho\colon G \to \mathrm{GL}(k, \mathbb{Q})$ be an irreducible representation of G.

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   1. $\rho = \operatorname{ind} \rho_H$, where $H < G, [G : H] = 2$ and $\rho_H : H \to \mathrm{GL}(k/2, \mathbb{Q})$:
      - $\rho_H \sim \varphi_H$, where $\varphi_H : H \to \mathrm{O}(k/2, \mathbb{Z})$.
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   2. $[G : \ker\rho] = 2$:

$$\rho(G) \subset \{\pm 1\} = O(1, \mathbb{Z}).$$

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Determining existence of spin structures

Let Γ be a Bieberbach group:

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1,$$

where $G \subset \mathrm{SO}(n)$, i.e. $\mathbb{R}^n/\Gamma$ is orientable.

1. Calculate a Sylow 2-subgroup F of G and deal with $\pi^{-1}(F) \subset \Gamma$. Let
$$\pi^{-1}(F) = \langle X \mid R \rangle$$
be a finite presentation of $\pi^{-1}(F)$.
2. Determine a representation $\varphi\colon F \to \mathrm{SO}(n, \mathbb{Z})$ equivalent to $id\colon F \to F \subset \mathrm{SO}(n)$.
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where $G \subset \mathrm{SO}(n)$, i.e. $\mathbb{R}^n/\Gamma$ is orientable.

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Spin structures in dimensions 5 and 6

dim	flat mflds	orientable f.m.	spin f.m.
5	1060	174	88
6	38746	3314	760

Thank you!