Holonomy groups of flat manifolds with R_∞ property

Rafał Lutowski
joint work with Andrzej Szczepański

Institute of Mathematics, University of Gdańsk

DMV & PTM Joint Meeting

17-20 September 2014

Outline

Fixed point theory

- X – closed connected triangulable manifold (global assumption).

$$f\colon X \to X$$

- $\mathrm{Fix}(f) := \{x \in X \mid f(x) = x\}$
- $\mathrm{MF}(f) := \min\{\#\mathrm{Fix}(h) \mid h \simeq f\}$
- $L(f)$ – Lefschetz number
- $R(f)$ – Reidemeister number
- $N(f)$ – Nielsen number

Lefschetz, Reidemeister and Nielsen numbers

Short summary

- All are homotopy invariants.
- $L(f) \neq 0 \Rightarrow \mathrm{MF}(f) > 0$.
- $N(f) \leq R(f)$.
- $R(f) \in \mathbb{Z}_+ \cup \{\infty\}$.
- $N(f) \in \mathbb{N}$.
- $N(f) \leq \mathrm{MF}(f)$.

Theorem 1 (Wecken 1942)

Let $f \colon X \to X$ be a self-map of a compact connected triangulated n-manifold X, where $n \geq 3$. Then there exists a map $g \simeq f$, such that $N(g) = \#\mathrm{Fix}(g)$.

Jiang-type spaces

Definition 1

A manifold X is a Jiang-type space if for every map $f\colon X \to X$

$$N(f) = \begin{cases} 0 & \text{if } L(f) = 0, \\ R(f) & \text{if } L(f) \neq 0. \end{cases}$$

Example 1

- Simply-connected spaces.
- Lens spaces.
- Nilmanifolds.

R_∞ property

Definition 2

A manifold X has the R_∞ property, if for every homeomorphism $f\colon X \to X$, $R(f) = \infty$.

Remark 1

1. The Reidemeister number (and thus the R_∞ property) can be defined at the level of fundamental group of X.
2. This definition can be easily applied to every countable discrete group.

Example 2

A class of groups with the R_∞ property includes:

- non-virtually cyclic Gromov hyperbolic groups (Fel'shtyn 2001),
- Baumslag-Solitar groups except from $BS(1,1)$ (Fel'shtyn 2008).

Bieberbach groups

- $\mathrm{Iso}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ – the group of isometries of $\mathbb{R}^n$:

$$\forall_{A \in O(n)} \forall_{a,x \in \mathbb{R}^n} (A, a) \cdot x = Ax + a.$$

- $\Gamma \subset \mathrm{Iso}(\mathbb{R}^n)$ – Bieberbach group:
  - discrete,
  - cocompact,
  - torsion-free.
- $X = \mathbb{R}^n / \Gamma$ – flat manifold (closed connected Riemannian n-manifold with zero sectional curvature).

Structure of Bieberbach groups

Theorem 2 (Bieberbach 1911)

The subgroup $\Gamma \cap (1 \times \mathbb{R}^n)$ of pure translations of Γ is free abelian group of rank n. Moreover it is maximal abelian normal subgroup of Γ of finite index.

- Γ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

- G – finite group – holonomy group of Γ (of X).
- Holonomy representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{Z})$ of Γ (of X) is defined as follows:

$$\forall_{g \in G} \forall_{m \in \mathbb{Z}^n}\ \varphi_g(m) = \overline{g} m \overline{g}^{-1},$$

where $p(\overline{g}) = g$.

Holonomy representation and R_∞-property

Remark 2 (Dekimpe, De Rock, Penninckx 2009)

The R_∞ property does not depend “directly” on the holonomy representation.

- Examples of manifolds with and without the R_∞ property, both with the same holonomy representation.
- If $f\colon X \to X$ is Anosov diffeomorphism, then $R(f) < \infty$.
  - Existence of Anosov diffeomorphisms is determined by the holonomy representation of X (Porteous 1972).

Multiplicity of image of representation

$\varphi\colon G \to \mathrm{GL}_n(\mathbb{Q}), \rho\colon G \to \mathrm{GL}_m(\mathbb{Q})$ – representations of a group G.

Definition 3

We say, that the image of the representation ρ is of multiplicity k in φ, if there exist exactly k subrepresntations $\rho_1, \ldots, \rho_k$ of φ such that

$$\forall_{1 \leq i \leq k} \exists_{C_i \in \mathrm{GL}_m(\mathbb{Q})} \; \rho(G) = C_i \rho_k(G) C_i^{-1}.$$

Remark 3

Multiplicity of an image of a representation is in general not the same as multiplicity of a representation ρ in φ. Those two numbers are equal in the case, when

$$N_{\mathrm{GL}_m(\mathbb{Q})}(\rho(G)) \twoheadrightarrow \mathrm{Aut}(G).$$

Holonomy representation and R_∞-property II

Theorem 3 (Dekimpe, De Rock, Penninckx 2009)

Let X be a flat manifold with holonomy representation

$$\varphi\colon G \to \mathrm{GL}_n(\mathbb{Z}).$$

Let

$$\rho\colon G \to \mathrm{GL}_m(\mathbb{Z})$$

be an irreducible $\mathbb{Q}$-subrepresentation of φ with the image of multiplicity 1. Suppose moreover that

$$\forall_{D \in N_{\mathrm{GL}_m(\mathbb{Z})}(\rho(G))} \exists_{g \in G} \ \det(I - \rho(g)D) = 0.$$

Then X has the R_∞ property.

Can we simplify/drop any assumption from the above theorem?

The conjecture

Conjecture 1 (Dekimpe, De Rock, Penninckx 2009)

Let $\rho\colon G \to \mathrm{GL}_m(\mathbb{Z})$ be a faithful $\mathbb{R}$-irreducible representation of a non-trivial finite group G. Suppose that m is odd. Then for every $D \in N_{\mathrm{GL}_m(\mathbb{Z})}(\rho(G))$, there exists $g \in G$ such that $\det(\mathrm{I} - \rho(g)D) = 0$.

- The conjecture cannot be generalized for any m (counterexample with $m = 4$).
- It is true for $m = 1, 3, 5$.

Conjecture 2

Let X be a flat manifold with holonomy representation $\varphi\colon G \to \mathrm{GL}_n(\mathbb{Z})$. Let $\rho\colon G \to \mathrm{GL}_m(\mathbb{Z})$ be an $\mathbb{R}$-irreducible $\mathbb{Q}$-subrepresentation of φ with an image of multiplicity 1 of odd degree. Then X has the R_∞ property.

Few facts and a question

$$\rho\colon G \to \mathrm{GL}_m(\mathbb{Z})$$

faithful ← · ↓ $\mathbb{Q}$ and $\mathbb{R}$-irreducible · → m – odd

- The normalizer $N_{\mathrm{GL}_m(\mathbb{Z})}(\rho(G))$ is finite (Szczepański 1996).
- Every odd degree matrix of finite order has an eigenvalue equal to ± 1 (characteristic polynomial).

Proposition 1

ρ is absolutely (i.e. $\mathbb{C}$-) irreducible.

Corollary 1

$$C_{\mathrm{GL}_m(\mathbb{Z})}(\rho(G)) = \langle -\mathrm{I} \rangle$$

- For groups with non-trivial center, the conjecture is true.
  - Ok for nilpotent groups

The theorem

Every solvable group has an abelian non-trivial normal subgroup.

Theorem 4

Let G be a finite group with a non-trivial normal abelian subgroup A and let $\rho\colon G \to \mathrm{GL}_m(\mathbb{Z})$ be a faithful $\mathbb{R}$-irreducible representation. Suppose m is odd. Then for every $D \in N_{\mathrm{GL}_m(\mathbb{Z})}(\rho(G))$, there exists $g \in G$ such that $\rho(g)D$ has eigenvalue 1.

Corollary 2

Let X be a flat manifold with a solvable holonomy group G and a holonomy representation $\varphi\colon G \to \mathrm{GL}_n(\mathbb{Z})$. Let $\rho\colon G \to \mathrm{GL}_m(\mathbb{Z})$ be an $\mathbb{R}$-irreducible $\mathbb{Q}$-subrepresentation of φ of odd degree, with the image of multiplicity 1. Then X has the R_∞ property.

The proof

Clifford's theorem

Theorem 5 (Clifford)

Let $\rho\colon G \to \mathrm{GL}_m(\mathbb{K})$ *be an irreducible representation of a group* G *over a field* $\mathbb{K}$*. Let* $H \lhd G$ *and* τ *be an irreducible component of* $\rho_{|H}$*. Then*

$$\rho_{|H} = e\tau^{(g_1)} \oplus \ldots \oplus e\tau^{(g_k)},$$

where $e \in \mathbb{Z}_+$ *is the multiplicity of* τ *in* $\rho_{|H}$*,* $g_1, \ldots, g_k \in G$ *and*

$$\forall_{1 \leq i \leq k} \forall_{a \in A}\ \tau^{(g_i)}(a) = \tau(g_i^{-1} a g_i).$$

Remark 4

In our case $\mathbb{K} = \mathbb{Q}$, $H = A$ is abelian. We get $m = ke\deg(\tau)$ (recall that m is odd).

The proof

Normal abelian subgroup

- $\rho_{|A} = e\tau^{(g_1)} \oplus \ldots \oplus e\tau^{(g_k)}$.
- $m = ke\deg(\tau)$.

Proposition 2

Let G be a finite group with normal nontrivial abelian subgroup A. Let $\rho\colon G \to \mathrm{GL}_m(\mathbb{Q})$ be an absolutely irreducible faithful representation of odd degree m. Then A is an elementary abelian 2-group.

$$\rho(A) \lhd \rho(G) \subset N_{\mathrm{GL}_m(\mathbb{Z})}(\rho(A))$$

- Elementary abelian 2-groups have very nice (diagonal) rational representations (with ± 1 on diagonal).
- For every $1 \leq i \leq k$ there exists $a \in A$, such that $\tau^{(g_i)}(a) = -1$.
- From now on we assume, that $\rho(A)$ is a group of diagonal matrices (proper basis).

The proof

The normalizer

$\rho_{|A} = e\tau^{(g_1)} \oplus \ldots \oplus e\tau^{(g_k)}$ – diagonal representation

1. The centralizer
$$
\begin{aligned}
C_{\mathrm{GL}_m(\mathbb{Q})}(\rho(A)) &= \{c_1 \oplus \ldots \oplus c_k \mid c_i \in \mathrm{GL}(e,\mathbb{Q}), i = 1,\ldots,k\} \\
&= \underbrace{\mathrm{GL}(e,\mathbb{Q}) \oplus \ldots \oplus \mathrm{GL}(e,\mathbb{Q})}_{k}.
\end{aligned}
$$
2. The normalizer. If $D \in N_{\mathrm{GL}_m(\mathbb{Q})}(\rho(A))$, then there exists $\sigma \in S_k$, such that
$$D(a_1 \oplus \ldots \oplus a_k)D^{-1} = a_{\sigma(1)} \oplus \ldots \oplus a_{\sigma(k)}$$
for every $a_1 \oplus \ldots \oplus a_k \in \rho(A)$.

Theorem 6

The normalizer $N := N_{\mathrm{GL}_m(\mathbb{Q})}(\rho(A))$ is isomorphic to a wreath product of $\mathrm{GL}_e(\mathbb{Q})$ and a subgroup S of S_k

The proof

Further properties of G

We have the following short exact sequence

$$1 \longrightarrow C \longrightarrow G \stackrel{p}{\longrightarrow} Q \longrightarrow 1.$$

- $C := C_G(A) = \rho^{-1}(C_{\mathrm{GL}_m(\mathbb{Q})}(\rho(G)))$.
- $Q \subset S \subset S_k$ and for every $\sigma \in Q$ we have

$$\rho(\bar{\sigma})(e\tau^{(g_1)} \oplus \ldots \oplus e\tau^{(g_k)})\rho(\bar{\sigma})^{-1} = e\tau^{(g_{\sigma(1)})} \oplus \ldots \oplus e\tau^{(g_{\sigma(k)})},$$

  where $p(\bar{\sigma}) = \sigma$.

Lemma 1

Q is a transitive group of permutations.

The proof

The uniqueness of A

$$1 \longrightarrow C \longrightarrow G \longrightarrow Q \longrightarrow 1$$

Additional assumption: A is a maximal (normal abelian) subgroup of G.

Lemma 2

A is unique in C.

Lemma 3

Q does not contain any non-trivial abelian normal 2-subgroup.

Proposition 3

A is a characteristic subgroup of G. Hence

$$N_{\mathrm{GL}_m(\mathbb{Q})}(\rho(G)) \subset N_{\mathrm{GL}_m(\mathbb{Q})}(\rho(A)).$$

The proof

The end

- Let $D \in N_{\mathrm{GL}_m(\mathbb{Q})}(\rho(G))$ be of finite order. It is of the form

$$D = P_\sigma(c_1 \oplus \ldots \oplus c_k),$$

  where P_σ – "block permutation matrix" corresponding to $\sigma \in S \subset S_k$.
- $Q = p(G)$ – transitive subgroup of S_k. Take $\pi \in Q$ such that

$$\pi(1) = \sigma^{-1}(1).$$

- Take any $\bar{\pi} \in p^{-1}(\pi)$. We get $\rho(\bar{\pi}) = (d_1 \oplus \ldots \oplus d_k)P_\pi$ and

$$(d_1 \oplus \ldots \oplus d_k)P_\pi P_\sigma(c_1 \oplus \ldots \oplus c_k) = \\ (d_1 \oplus \ldots \oplus d_k)P_{\sigma\pi}(c_1 \oplus \ldots \oplus c_k) = d_1c_1 \oplus X.$$

- Pick $a \in A$, such that $\tau^{(g_1)}(a)d_1c_1$ has eigenvalue 1.

$$\det(\mathrm{I} - \rho(a\bar{\pi})D) = 0.$$

□

Questions remaining

- The conjecture is true for „small" class of finite groups. What about other groups (e.g. finite simple groups)?
- The conjecture cannot be generalized to all integers, but maybe we can simplify the theorem without assuming m being odd?

Simple alternating groups

Proposition 4

Let $\rho\colon A_n \to \mathrm{GL}_m(\mathbb{Z})$ *be an* $\mathbb{R}$*-irreducible representation of a simple alternating group* A_n *(*$n \geq 5$*). Let* m *be odd. Then for every* $D \in N := N_{\mathrm{GL}_m(\mathbb{Z})}(\rho(A_n))$ *there exists* $g \in A_n$ *such that* $\rho(g)D$ *has an eigenvalue* 1*.*

1. A_6 – by computer.
2. For $n \neq 6, n \geq 5$ we have $\mathrm{Aut}(A_n) \cong S_n \cong A_n \rtimes C_2$.
3. Let $D \in N$ st. D^2 induces the identity automorphism of A_n:

$$D^2 \in C_{\mathrm{GL}_m(\mathbb{Z})}(\rho(A_n)) = \{\pm \mathrm{I}\}.$$

4. D has an eigenvalue ± 1, hence $D^2 = \mathrm{I}$.
5. $D \neq -\mathrm{I} \Rightarrow \det(\mathrm{I} - D) = 0$.
6. If $D = -\mathrm{I}$ take any $g \in A_n$ of order 2. Then $\det(\mathrm{I} - \rho(g)D) = 0$.

Thank you!