

MODULAR REPRESENTATION THEORY

(VERY) SHORT INTRODUCTION

RAFA LUTOWSKI

UNIVERSITY OF GDASK

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COMPLETE FIELDS

p -ADIC NORM

Global assumption

p – a fixed prime

p -adic norm

$a = p^k \cdot m/n \in \mathbb{Q}, k \in \mathbb{Z}, p \nmid mn:$

$$|a|_p := p^{-k}$$

Properties of p -adic norm

1. $|a|_p = 0 \Leftrightarrow a = 0.$
2. $|ab|_p = |a|_p |b|_p.$
3. $|a + b|_p \leq \max(|a|_p, |b|_p).$

COMPLETION OF \mathbb{Q}

Series ($n \in \mathbb{Z}, 0 \leq a_i < p$)

$$\sum_{i=n}^{\infty} a_i p^i$$

Partial sums

$$s_k = \sum_{i=n}^k a_i p^i$$

(s_k) is a Cauchy sequence

$$\lim_{k \rightarrow \infty} |s_k - s_{k-1}|_p = \lim_{k \rightarrow \infty} |a_k p^k|_p = 0$$

Definition

The completion of \mathbb{Q} – the field \mathbb{Q}_p of p -adic numbers.

p -ADIC NUMBERS

Notation

$$x = \sum_{i=n}^{\infty} a_i p^i = [\dots a_3 a_2 a_1 a_0 . a_{-1} \dots a_n]_p$$

Note: $|x|_p = p^{-n}$.

2-adic numbers

$$3 = 11_2 \quad 2.25 = 10.01_2 \quad \dots 0101011_2 = (01)_2 = \frac{1}{3}$$

p -adic integers

$$\mathbb{Z}_p := \{q \in \mathbb{Q}_p : |q| \leq 1\}$$

p -ADIC INTEGERS

PID

Ideals: (p^k) , $k \in \mathbb{N}$.

Local ring

Maximal ideal: (p) .

Valuation ring

$\mathbb{Q}_p = \mathbb{Z}_p \cup \mathbb{Z}_p^{-1}$

\mathbb{Z}_p is PID + \mathbb{Z}_p is valuation + \mathbb{Q}_p is complete

\mathbb{Z}_p – complete discrete valuation ring

Residue field

$\mathbb{Z}_p/(p) \cong \mathbb{F}_p$.

mod p reduction

$\mathbb{Z}_p \rightarrow \mathbb{Z}_p/(p)$, $a \mapsto \bar{a}$.

FINITE EXTENSIONS OF \mathbb{Q}_p

F finite extension of \mathbb{Q}_p

$$a \in F, n = (F : \mathbb{Q}_p)$$

L_a – left multiplication by a

$$L_a \in \text{Hom}_{\mathbb{Q}_p}(F, F)$$

p -adic norm extension

$$|a|_p = |\det L_a|_p^{1/n}$$

Integral elements of F

$$\mathcal{O}_F := \{a \in F : |a|_p \leq 1\}$$

INTEGRAL ELEMENTS OF F

PID

Ideals: $(\pi^k), k \in \mathbb{N}$.

Local ring

Maximal ideal: (π) .

Valuation ring

$F = \mathcal{O}_F \cup \mathcal{O}_F^{-1}$

\mathcal{O}_F is PID + \mathcal{O}_F is valuation + F is complete

\mathcal{O}_F - **complete discrete valuation ring**

Residue field

$\bar{F} = \mathcal{O}_F/(\pi) \cong \mathbb{F}_q$

mod π reduction

$\mathcal{O}_F \rightarrow \mathcal{O}_F/(\pi), a \mapsto \bar{a}$

FG , $\mathcal{O}_F G$ AND $\overline{F}G$ -MODULES

BURNSIDE THEOREM

$\mathcal{O}_F G$ -lattice

$\mathcal{O}_F G$ -module, free and of finite rank as \mathcal{O}_F -module

Theorem

Let V be an FG -module of finite dimension n over F . There exists an $\mathcal{O}_F G$ -lattice $W \subset V$, of \mathcal{O}_F -rank n , st. $F \otimes_{\mathcal{O}_F} W = V$.

$F = \mathbb{Q}_2$, $G = C_2 = \langle c \rangle$, $V = FG$ - regular module

$$W_1 = \langle 1, c \rangle, W_2 = \langle (1+c)/2, (1-c)/2 \rangle.$$

Basis of W is a basis of V

$$\chi_W(g) = \chi_V(g) \text{ for every } g \in G$$

REDUCTION

\mathcal{O}_F -module \overline{F} . $a \in \mathcal{O}_F, u \in \overline{F}$:

$$\text{(left)} \quad a \cdot u := \overline{a}u = u\overline{a} =: u \cdot a \quad \text{(right)}$$

$W - \mathcal{O}_F G$ -lattice, $\overline{W} := \overline{F} \otimes_{\mathcal{O}_F} W$

Mapping $W \rightarrow \overline{W}$, given by $w \mapsto \overline{w} = \overline{1} \otimes w$. Module structure

$$\overline{a} \cdot \overline{w} = \overline{a} \otimes w = \overline{1} \otimes aw = \overline{a}\overline{w}$$

$W - \mathcal{O}_F G$ -lattice, $\overline{W} := \overline{F} \otimes_{\mathcal{O}_F} W, V = F \otimes_{\mathcal{O}_F} W$:

$$\chi_{\overline{W}}(g) = \overline{\chi_W(g)} = \overline{\chi_V(g)} \text{ for every } g \in G.$$

IDEMPOTENTS IN RINGS

Idempotent

$$e^2 = e \neq 0$$

Orthogonal idempotents

$$e_1 e_2 = e_2 e_1 = 0$$

(Central) Primitive idempotent:

not a sum of (central) orthogonal idempotents

DECOMPOSITION

e_1, \dots, e_n – primitive orthogonal idempotents of a ring R

$$1 = e_1 + \dots + e_n \leftrightarrow R = I_1 \oplus \dots \oplus I_n$$

$I_k = Re_k$ – indecomposable left ideal (left R -module)

Decomposition of a left R -module W

$$W = e_1 W \oplus \dots \oplus e_n W$$

c_1, \dots, c_h – primitive central orthogonal idempotents of R

$$1 = c_1 + \dots + c_h \leftrightarrow R = B_1 \oplus \dots \oplus B_h$$

$B_k = Rc_k = c_k R$ – indecomposable (two-sided) ideal (block)

DECOMPOSITION: FG VS. $\mathcal{O}_F G$

$$F = \mathbb{Q}_2, G = C_2 = \langle c \rangle, \mathcal{O}_F = \mathbb{Z}_2$$

Idempotents in FG :

$$1, e_1 = (1 + c)/2, e_2 = (1 - c)/2$$

$1 = e_1 + e_2, e_1 e_2 = 0$, but $e_1, e_2 \notin \mathcal{O}_F G$.

REDUCTION/LIFTING OF IDEMPOTENTS

Theorem

Every decomposition of 1 of $\overline{F}G$ to the sum of pairwise orthogonal primitive (central primitive) idempotents

$$1 = u_1 + u_2 + \dots + u_m$$

corresponds to a decomposition of 1 of $\mathcal{O}_F G$ to the sum of pairwise orthogonal primitive (central primitive) idempotents

$$1 = e_1 + e_2 + \dots + e_m$$

satisfying $\overline{e_i} = u_i$, for $i = 1, \dots, m$.

Lift may not be unique: $u \in (\mathcal{O}_F G)^*$, $\overline{u} = 1$

$$1 = ue_1u^{-1} + ue_2u^{-1} + \dots + ue_mu^{-1}$$

REDUCTION/LIFTING OF IDEMPOTENTS

Theorem

$$\mathcal{O}_F G \ni 1 = e_1 + e_2 + \dots + e_m \quad \leftrightarrow \quad u_1 + u_2 + \dots + u_m = 1 \in \overline{F}G$$

Corollary

$$\mathcal{O}_F G = W_1 \oplus \dots \oplus W_r \quad \leftrightarrow \quad \overline{W}_1 \oplus \dots \oplus \overline{W}_r = \overline{F}G$$

W_i, \overline{W}_i - indecomposable, for $i = 1, \dots, r$

Corollary

$$\mathcal{O}_F G = B_1 \oplus \dots \oplus B_t \quad \leftrightarrow \quad \overline{B}_1 \oplus \dots \oplus \overline{B}_t = \overline{F}G$$

B_i, \overline{B}_i - blocks, for $i = 1, \dots, t$

REDUCTION/LIFTING OF IDEMPOTENTS: EXAMPLE

$$F = \mathbb{Q}_2, G = S_3$$

$$a = (1\ 2\ 3), b = (1\ 2)$$

$$d = 1 + a + a^2$$

$$d \in Z(\mathcal{O}_F G), d^2 = 3d$$

$$c_1 = d/3, c_2 = 1 - c_1$$

$$1 = c_1 + c_2$$

Primitive orthogonal idempotents of FG ($c = c_1$)

$$\begin{aligned} e_1 &= (1 + b)c/2, & e_2 &= (1 - b)c/2 & : & e_1 + e_2 = c_1 \text{ in } FG \\ e_3 &= (c - a^2)(1 + b), & e_4 &= (1 - b)(c - a) & : & e_3 + e_4 = c_2 \text{ in } \mathcal{O}_F G \end{aligned}$$

Primitive orthogonal idempotents of $\overline{F}G$

$$\bar{c} = 1 + \bar{a} + \bar{a}^2, \bar{e}_3 = (1 + \bar{a})(1 + \bar{b}), \bar{e}_4 = (1 + \bar{b})(1 + \bar{a}^2)$$

BRAUER CHARACTERS

ASSUMPTIONS, DEFINITIONS

F – finite extension of \mathbb{Q}_p + splitting field for G

F – finite extension of splitting field for $x^{\exp G} - 1$ in \mathbb{Q}_p

Corollary

\overline{F} – finite extension of \mathbb{F}_p + splitting field for G

$\zeta \in \mu_{p^k}$ – primitive m -th root

ζ – primitive p^k -th root

$\overline{\zeta}$ – primitive m -th root

$\zeta = 1 + (\zeta - 1), |\zeta - 1|_p < 1 \Rightarrow \overline{\zeta} = 1$

Corollary

$\overline{\cdot} : \mu_{p^k} \rightarrow \overline{\mu}_{p^k}$ – bijection

IMPORTANT CASE

$H = \langle a \rangle$ – cyclic of order m prime to p

ζ – primitive m -th root of 1 $\in FH$:

$$1 = e_1 + e_\zeta + e_{\zeta^2} + \dots + e_{\zeta^{m-1}}$$

where $ae_x = xe_x$ for $x = 1, \zeta, \dots, \zeta^{m-1}$:

$$e_x = (1 + x^{-1}a + x^{-2}a^2 + \dots + x^{-m+1}a^{m-1})/m \in \mathcal{O}_F H.$$

$e_1, \dots, e_{\zeta^{m-1}}$ – primitive orthogonal and central idempotents.

Over $\mathcal{O}_F H$ and \overline{FH}

1. indecomposable module = irreducible module
2. indecomposable = of rank/dimension 1

BRAUER CHARACTER

\overline{W} – $\overline{F}G$ -module

Goal: define $\varphi_{\overline{W}}: G_{p'} \rightarrow \mathcal{O}_F$

$\text{res}_H \overline{W} = \overline{W}_1 \oplus \dots \oplus \overline{W}_r$

$$\forall_i \exists x_i \in \mu_{p'} \forall w \in \overline{W}_i g \cdot w = \overline{x}_i w$$

g of order m prime to p , $H = \langle g \rangle$

$\zeta \in \mu_{p'}$ – primitive m -th root

Lift of $\text{res}_H \overline{W}$: $\mathcal{O}_F H$ -lattice W

$$W = W_1 \oplus \dots \oplus W_r$$

Brauer character

$$\varphi_{\overline{W}}(g) := \chi_W(g) = x_1 + \dots + x_r$$

\overline{W} – simple: $\varphi_{\overline{W}}$ – simple

$\text{IBr}_p G$ – set of Brauer characters of simple $\overline{F}G$ -modules

EXAMPLE

$$F = \mathbb{Q}_2(\zeta), \quad \zeta^2 + \zeta + 1 = 0, \quad \mathcal{O}_F = \mathbb{Z}_2[\zeta], \quad \overline{F} = \mathbb{F}_2(\overline{\zeta}), \quad G = S_3 = \langle a, b \rangle$$

$$\text{Simple } \overline{F}G\text{-module } \overline{W} = \overline{F}^2, \quad a \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues of a and a^2 : $\overline{\zeta}, \overline{\zeta}^2$

$$\chi_{\overline{W}}(1) = 0 \text{ and } \chi_{\overline{W}}(a) = \chi_{\overline{W}}(a^2) = \overline{\zeta} + \overline{\zeta}^2 = 1$$

but

$$\varphi_{\overline{W}}(1) = 2 \text{ and } \varphi_{\overline{W}}(a) = \varphi_{\overline{W}}(a^2) = \zeta + \zeta^2 = -1$$

BRAUER CHARACTERS: PROPERTIES

Lemma

- *Every sum of Brauer characters is a Brauer character.*
- *Every Brauer character is of the form*

$$\psi = \sum_{\varphi \in \text{IBr}_p G} n_\varphi \varphi$$

where $n_\varphi \in \mathbb{Z}_{\geq 0}$.

- *Every Brauer character is constant on every p -regular conjugacy class.*

BRAUER CHARACTERS: PROPERTIES

$$g \in G, |g| = qm, q = p^k, p \nmid m$$

$$\exists_{k,l \in \mathbb{Z}} km + lq = 1$$

$$g_{p'} = g^{lq}$$

p -regular g -part

$$g_p = g^{km}$$

p -singular g -part

Lemma

Let \overline{W} be an $\overline{F}G$ -module, $\chi_{\overline{W}}$ and $\varphi_{\overline{W}}$ – its character and Brauer character respectively. Then $\chi_{\overline{W}}(g) = \varphi_{\overline{W}}(g_{p'})$.

BRAUER CHARACTERS: PROPERTIES

Theorem

1. Brauer characters corresponding to non-isomorphic $\overline{F}G$ -simple modules are linearly independent.
2. The number of simple Brauer characters is equal to the number of p -regular conjugacy classes of G .

Corollary

The set of the simple Brauer characters is a basis of the space of functions which are constant on each conjugacy class of p -regular elements of G .

Theorem

If $p \nmid |G|$ then $\text{IBr}_p G = \text{Irr } G$.

EXAMPLE: BRAUER TABLE FOR S_3

Step 1: Ordinary character table

	$()$	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Step 2: Drop 2-singular classes

	$()$	(123)
χ_1	1	1
χ_2	1	1
χ_3	2	-1

Step 3: Brauer character table for $p = 2$

	$()$	(123)
φ_1	1	1
φ_2	2	-1

CHARACTERS IN BLOCKS

Block decomposition

$$\mathcal{O}_F G = B_1 + \dots + B_r$$

$B \in \{B_i\}$

$$B = \mathcal{O}_F G c$$

FG -simple V

$$V \in B: cV = V$$

$v \in V$

$$c(cv) = cv$$

$$\chi_V(c) = \dim V$$

$$V \in B \Leftrightarrow \chi_V(c) \neq 0$$

Characters in block

$$\text{Irr } B := \{\chi \in \text{Irr } G : \chi(c) \neq 0\}$$

Example: $\mathbb{Q}_2 S_3$

$$\chi_1, \chi_2 \in B_1, \chi_3 \in B_2$$

DEFECTS OF CHARACTERS

Valuation $\nu_\pi: FG \rightarrow \mathbb{Z}$

$$\nu_\pi(x) = n \Leftrightarrow x = \pi^n y, y \in \mathcal{O}_F^*$$

p -defect of character $\chi \in \text{Irr } G$

$$d_p(\chi) = \nu_p(|G|) - \nu_p(\chi(1))$$

B - block, $\chi \in \text{Irr } B$

$$\underbrace{d_p(\chi) = 0 \Leftrightarrow \text{Irr } B = \{\chi\} \Leftrightarrow \chi \text{ - char. of proj. } \mathcal{O}_F G\text{-mod.}}_{\Downarrow}$$

$\chi|_{G_{p'}}$ - simple Brauer character of G

DEFECTS: APPLICATION FOR $G = S_4, p = 3$

Convention

$$1a = ()^G, 2a = (12)(34)^G, 3a = (123)^G, 2b = (12)^G, 4a = (1234)^G$$

1: Ordinary character table

	1a	2a	3a	2b	4a
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	1	-1
χ_5	3	-1	0	-1	1

2: Drop 3-singular classes

	1a	2a	2b	4a
χ'_1	1	1	1	1
χ'_2	1	1	-1	-1
χ'_3	2	2	0	0
χ'_4	3	-1	1	-1
χ'_5	3	-1	-1	1

DEFECTS: APPLICATION FOR $G = S_4, p = 3$

Convention

$$1a = ()^G, 2a = (12)(34)^G, 3a = (123)^G, 2b = (12)^G, 4a = (1234)^G$$

$$3: \chi'_3 = \chi'_1 + \chi'_2$$

	$1a$	$2a$	$2b$	$4a$
χ'_1	1	1	1	1
χ'_2	1	1	-1	-1
χ'_4	3	-1	1	-1
χ'_5	3	-1	-1	1

DEFECTS: APPLICATION FOR $G = S_4, p = 3$

Convention

$$1a = ()^G, 2a = (12)(34)^G, 3a = (123)^G, 2b = (12)^G, 4a = (1234)^G$$

$$3: \chi'_3 = \chi'_1 + \chi'_2$$

	$1a$	$2a$	$2b$	$4a$
φ_1	1	1	1	1
φ_2	1	1	-1	-1
φ_3	3	-1	1	-1
φ_4	3	-1	-1	1

4: $d_3(\chi_3) = d_3(\chi_4) = 0$ – We've got Brauer character table