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LOOPS'23

Minimal non-solvable Bieberbach groups

Math databases matter

with Andrzej Szczepański

Main question

Question (Hillman 2022)

What is a minimal Hirsch length $h(\Gamma)$ *of a torsion-free virtually polycyclic non-solvable group* Γ*?*

Three cases

Theorem (Hillman 2023)

1. Γ – *virtually solvable (general case):*

$$h(\Gamma) \geq 10.$$

2. Γ – *virtually nilpotent and* $h(\Gamma) \leq 14$*, then the Fitting subgroup is of nilpotency class* ≤ 3.

Theorem (Lutowski, Szczepański 2023)

3. Γ – *virtually abelian of minimal Hirsch length:*

$$h(\Gamma) = 15.$$

Torsion-free virtually abelian (of finite rank) groups

Biebierbach groups

Torsion-free virtually abelian (of finite rank) groups

Biebierbach groups

$$0 \longrightarrow L \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

$L = \left\{ z \in \mathbb{Q}^n : \begin{bmatrix} I & z \\ 0 & 1 \end{bmatrix} \in \Gamma \right\}$, $L \cong \mathbb{Z}^n$

$s : G \to \mathbb{Q}^n$

$G \subset \mathrm{GL}(n, \mathbb{Z})$ – finite

$$\Gamma = \left\{ \begin{bmatrix} g & s(g) + z \\ 0 & 1 \end{bmatrix} : g \in G, z \in L \right\}$$

Digression 1: YBE

- $X = \{1, \dots, n\}$ for some $n \in \mathbb{N}$
- $r(x, y) = (\sigma_x(y), \tau_y(x))$
- (X, r) – involutive non-degenerate set-theoretic solution
- Structure group of the solution:

$$G(X, r) := \langle X \mid xy = uv \text{ if } r(x, y) = (u, v) \rangle$$

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Theorem (Gateva-Ivanova, Van den Bergh 1998)

$G(X, r)$ is a Bieberbach group.

Theorem (Etingof, Schedler, Soloviev 1999; Acri, Lutowski, Vendramin 2020)

There exists an injective group homomorphism $G(X, r) \to \mathrm{GL}_{n+1}(\mathbb{Z})$ given by

$$x \mapsto \begin{bmatrix} \sigma_x & t_x \\ 0 & 1 \end{bmatrix}.$$

Digression 2: left brace structure

Define $\pi\colon \Gamma \to \mathbb{Q}^n$ by

$$\pi\left(\begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix}\right) = a.$$

For $\Gamma = G(X, r)$, π is injective and $\pi(\Gamma)$ is a group:

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Remark

1. For Bieberbach groups c is always injective.
2. In general $\pi(\Gamma)$ is not a group, but it is enough to check if $\pi(\Gamma)/L$ is one.

Example: Promislow group

$$x = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, y = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$\Gamma = \langle x, y \rangle$ – Promislow group and $L = \mathbb{Z}^n$.

s	$\pi(s) \bmod L$
x	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$
y	$\left(0, \frac{1}{2}, \frac{1}{2}\right)$
xy	$\left(\frac{1}{2}, 0, \frac{1}{2}\right)$

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y	$\left(0, \frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{2}\right)$
xy	$\left(\frac{1}{2}, 0, \frac{1}{2}\right)$	$\left(\frac{1}{4}, 0, \frac{1}{2}\right)$

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xy	$(\frac{1}{2}, 0, \frac{1}{2})$	$(\frac{1}{4}, 0, \frac{1}{2})$

Corollary

Γ is a left brace, while Γ^z is not.

Datum of a Bieberbach group

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

1. $G \subset \mathrm{GL}(n, \mathbb{Z})$ – finite.
2. $\alpha = [\overline{s}] \in H^1(G, \mathbb{Q}^n/\mathbb{Z}^n)$ – a special cohomology class, where
$$\overline{s}(g) = s(g) + \mathbb{Z}^n.$$

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Remark

Solvability of Γ is built into G:

$$\Gamma \text{ is solvable } \Leftrightarrow G \text{ is solvable}$$

Minimal non-solvable Bieberbach groups

Definition

Let Γ be a Bieberbach group as above. We will call Γ *minimal non-solvable* (*MNS*), if every subgroup Γ' of Γ such that

- Γ' is of smaller dimension than Γ or
- $\Gamma' = \pi^{-1}(H)$ for some proper subgroup H of G

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Theorem (Hiller-Marciniak-Sah-Szczepański 1987, Plesken 1989)

There exists a non-solvable Bieberbach group of dimension 15.

Proposition

If Γ is a MNS Bieberbach group, then $h(\Gamma) \geq 15$.

Holonomy of MNS Bieberbach groups

Proposition

If $G \subset \mathrm{GL}(n, \mathbb{Z})$ is a holonomy group of a MNS Bieberbach group, then

1. *G is perfect.*
2. *All maximal subgroups of G are solvable (G is MNS in the usual sense).*
3. *The action of G on $\mathbb{Z}^n$ ($\mathbb{Q}^n, \mathbb{C}^n$) has certain properties (black box for this talk).*

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$> 10^6$: by 1 check in the GAP library of finite prefect groups of order $\leq 10^6$ and then do check as above with lower bound 10^6 for the order of the group.

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Maximal orders finite irreducible subgroups of $\mathrm{GL}(n,\mathbb{Z})$

dimension	4	5	6	7	8	9	10
max order	$1.1 \cdot 10^3$	$3.8 \cdot 10^3$	$0.1 \cdot 10^6$	$2.9 \cdot 10^6$	$0.7 \cdot 10^9$	$0.2 \cdot 10^9$	$3.7 \cdot 10^9$

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3. The subgroups were calculated up to conjugacy in the whole group:

```
# Approach "All" and [Approach ">10^6"]:

MaximalNonsolvableSubgroups := function(grp[, min])
    return Filtered(
        MaximalSubgroupClassReps(grp),
        x -> [Size(x)>=min and] not IsSolvableGroup(x) );
end;
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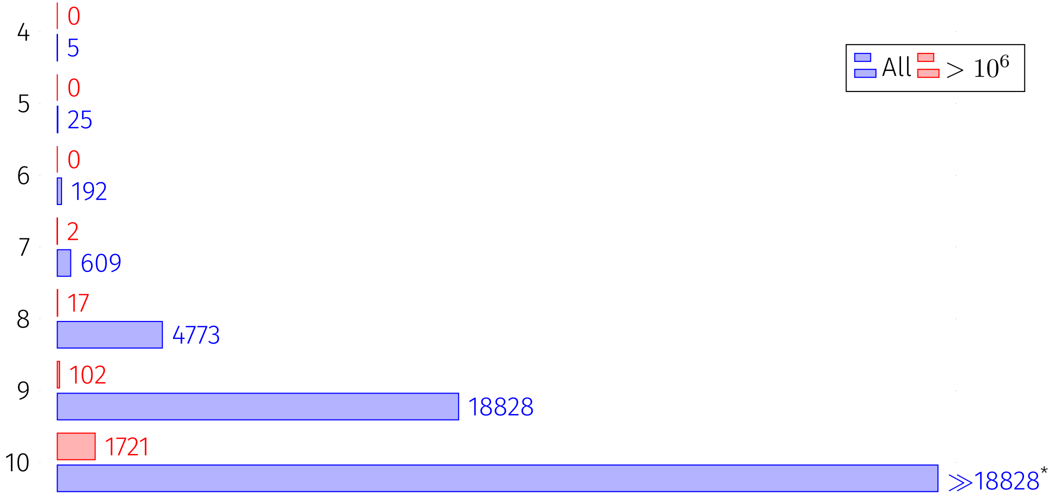
4. Thanks to the library, we didn't have to work with matrix groups, e.g.

```
gap> G := Image( IsomorphismPermGroup( ImfMatrixGroup(10,1,1) ) );
```

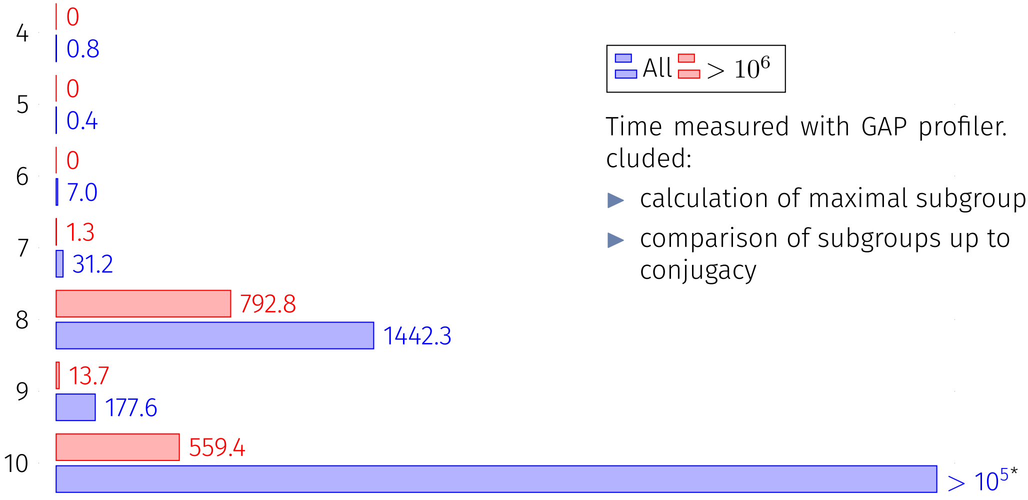
All images lie in S_n for $n \leq 270$ (median for n is 42, average ≈ 63).

Count of maximal subgroups calculation



*ran out of memory

Runtime (in seconds)



Time measured with GAP profiler. Included:

- calculation of maximal subgroups
- comparison of subgroups up to conjugacy

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Some remarks

- Calculation of perfect groups of order $< 10^6$ which do not have proper non-solvable subgroups took about 1077 seconds.

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Conclusion of calculations

Every subgroup of a finite irreducible subgroup of $\mathrm{GL}(n, \mathbb{Z})$ and of order $> 10^6$ has a non-solvable subgroup.

Thank you!