Irreducible euclidean representations of the Fibonacci groups

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Euclidean and affine maps in $\mathbb{R}^n$

- $E(n) = \mathrm{Iso}(\mathbb{R}^n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$ – the group of isometries of the Euclidean space $\mathbb{R}^n$.
- $A(n) = \mathrm{Aff}(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ – the group of affine maps of $\mathbb{R}^n$.

Remark

1. $E(n) \subset A(n)$.
2. $A(n) = \{(A, a) \mid A \in \mathrm{GL}(n, \mathbb{R}), a \in \mathbb{R}^n\}$ and
$$(A, a)(B, b) = (AB, Ab + a)$$
for $(A, a), (B, b) \in A(n)$.
3. The action of the group $A(n)$ ($E(n)$) on $\mathbb{R}^n$:
$$(A, a) \cdot x = Ax + a$$
for $(A, a) \in A(n), x \in \mathbb{R}^n$.

Euclidean and affine maps in $\mathbb{R}^n$

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Lemma

There is a faithful representation $A(n) \to \mathrm{GL}_{n+1}(\mathbb{R})$ *given by*

$$\forall_{(A,a)\in A(n)}\ (A, a) \mapsto \begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix}.$$

Crystallographic and Bieberbach groups

Definition

A group Γ is an n-dimensional crystallographic group if it is a discrete and cocompact subgroup of $E(n)$. If in addition Γ is torsionfree then we call it a Bieberbach group.

Remark

If $\Gamma \subset E(n)$ is a Bieberbach group then $X = \mathbb{R}^n/\Gamma$ is a flat manifold. The affine equivalence class of X is fully determined by the isomorphism class of Γ.

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Bieberbach theorems

Theorem (Bieberbach 1911)

1. *Let $\Gamma \subset E(n)$ be an n-dimensional crystallographic group. The subgroup $\Gamma \cap (1 \times \mathbb{R}^n)$ of pure translations of Γ is free abelian group of rank n. Moreover it is maximal abelian normal subgroup of Γ of finite index.*

- Γ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$

- G – finite group – holonomy group of Γ.
- We get an integral holonomy representation $\varphi\colon G \to \mathrm{GL}_n(\mathbb{Z})$:

$$\varphi_g(z) = \gamma z \gamma^{-1},$$

where $g \in G, z \in \mathbb{Z}^n$ and $\gamma \in \Gamma$ st. $\pi(\gamma) = g$.

Hantzsche-Wendt groups

Example (Classical Hantzsche-Wendt group)

$$\Gamma = \left\langle \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right) \right\rangle \subset E(3)$$

Definition

Let $n \in \mathbb{N}$ be odd. Let $\Gamma \subset E(n)$ be a Bieberbach group with holonomy group G and holonomy representation $\varphi\colon G \to \mathrm{GL}_n(\mathbb{Z})$. If

$$G \cong C_2^{n-1} \quad \text{and} \quad \varphi(G) \subset \mathrm{SL}_n(\mathbb{Z})$$

then Γ is called a Hantzsche-Wendt group (HW-group) and $X = \mathbb{R}^n/\Gamma$ – a Hantzsche-Wendt manifold.

Structure of HW-groups

Theorem (Rosetti, Szczepański 2005)

Let $n \in \mathbb{N}$ be odd. Let Γ be an n dimensional HW-group. Then

$$\Gamma \cong \langle (B_i, b_i) \mid i = 1, \dots, n \rangle \subset E(n),$$

where

$$B_i = \operatorname{diag}(\underbrace{-1, \dots, -1}_{i-1}, 1, -1, \dots, -1)$$

and $b_i \in \frac{1}{2}\mathbb{Z}^n$ for $i = 1, \dots, n$.

Remark

The group $\langle (B_i, b_i) \mid i = 1, \dots, n \rangle$ is a Bieberbach group, hence it is a HW-group.

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Encoding HW-groups

- $\Gamma \subset E(n)$ – HW-group.
- $\Gamma \cong \langle (B_i, b_i) \mid i = 1, \ldots, n \rangle$.

Corollary

Up to isomorphism every HW-group is defined by a matrix

$$\begin{bmatrix} b_1 & \ldots & b_n \end{bmatrix} \in M_n(\tfrac{1}{2}\mathbb{Z}).$$

Fibonacci groups

Definition

Let $r, n \in \mathbb{N}$. The Fibonacci group $F(r, n)$ is a group with presentation

$$F(r,n) = \langle a_0, a_1, \ldots, a_{n-1} \mid a_0 a_1 \ldots a_{r-1} = a_r, \\ a_1 a_2 \ldots a_r = a_{r+1}, \\ \vdots \\ a_{n-1} a_0 \ldots a_{r-2} = a_{r-1} \rangle.$$

Global assumption

The subscripts will always be taken modulo n.

Proposition

Let Γ be the classical 3-dimensional HW-group. Then $\Gamma \cong F(2, 6)$.

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Fibonacci and Hantzsche-Wendt groups

Theorem (Szczepański 2001)

Let $n \in \mathbb{N}$ *be odd,* $n \geq 3$*. Let* $\Gamma \subset E(n)$ *be a HW-group defined by the matrix*

$$\begin{bmatrix} \frac{1}{2} & & & & & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & & & & \\ & \frac{1}{2} & \ddots & & & \\ & & \ddots & \frac{1}{2} & & \\ & & & \frac{1}{2} & \frac{1}{2} & \\ & & & & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Then there exists an epimorphism

$$\Phi\colon F(n-1, 2n) \to \Gamma.$$

Cyclic HW-groups

Let $\Gamma \subset E(n)$ be as in the previous theorem.

- Γ is "cyclic" because of the form of the matrix which defines it.
- Γ is "cyclic" because it has generators which behave exactly as the generators of some Fibonacci group.

Question

Is every HW-group cyclic in the second sense?

More precisely:

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Euclidean representation

Definition

An euclidean representation of a group G is any homomorphism $\varphi\colon G \to E(n)$ for some $n \in \mathbb{N}$.

Example

Let Γ be the n-dimensional cyclic HW-group. Then

$$\Phi\colon F(n-1,2n) \to \Gamma \subset E(n)$$

is an euclidean representation of the Fibonacci group $F(n-1,2n)$.

Decomposability and irreducibility

Example

Let $G = \langle a, b \mid [a,b] = 1 \rangle$ be a free abelian group of rank 2. Let $\varphi_1, \varphi_2 \colon G \to E(1) = \{\pm 1\} \ltimes \mathbb{R}$ be euclidean representations of the group G defined by

$$\varphi_1(a) = (1,1), \quad \varphi_1(b) = (1,0),$$
$$\varphi_2(a) = (1,0), \quad \varphi_1(b) = (1,1).$$

We get an euclidean representation $\varphi_1 \oplus \varphi_2 \colon G \to E(2)$:

$$\varphi_1 \oplus \varphi_2(a) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \quad \varphi_1 \oplus \varphi_2(b) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

The above action of G on $V = \mathbb{R} \oplus \mathbb{R}$ is defined as a direct sum, but V does not have proper invariant subspace under this action!

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Decomposition of euclidean representations

- $n \in \mathbb{N}$.
- $\pi\colon E(n) \to \mathrm{O}(n)$ – given by $\pi(B, b) = B$, $(B, b) \in E(n)$.
- $\varphi\colon G \to E(n)$ – euclidean representation of a group G.

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Decomposition of euclidean representations

$$G \xrightarrow{\varphi} E(n) \xrightarrow{\pi} \mathrm{O}(n)$$

- $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k$ – decomposition of $\pi\varphi \colon G \to \mathrm{O}(n)$.
- $V_1, \ldots, V_k$ – irreducible subrepresentations of $\pi\varphi$.
- $p_i \colon \mathbb{R}^n \to V_i$ – projections, $i = 1, \ldots, k$.
- For every $i = 1, \ldots, k$ we define $\varphi^{(i)} \colon G \to \mathrm{Iso}(V_i)$:

$$\varphi^{(i)}_g(v) = (A, p_i(a))v = Av + p_i(a),$$

where $g \in G, v \in V_i$ and $\varphi_g = (A, a)$.

Proposition

$$\varphi = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(k)}$$

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Decomposition of HW-groups

Corollary

Let $n \in \mathbb{N}$ be odd. Let $\Gamma \subset E(n)$ be an n-dimensional HW-group defined by a matrix $[b_{ij}]_{1 \leq i,j \leq n}$. Then

$$id_\Gamma = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(n)},$$

where homomorphisms $\varphi^{(i)} \colon \Gamma \to E(1)$ are given by

$$\forall_{1 \leq j \leq n} \; \varphi^{(i)}(B_j, b_j) = \left((-1)^{\delta_{ij}+1}, b_{ij} \right).$$

Decomposition of HW-groups

$$id_\Gamma = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(n)}$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \right)$$

$$\shortparallel$$

$$(-1, b_{1,j}) \oplus \ldots \oplus (-1, b_{j-1,j}) \oplus (1, b_{j,j}) \oplus (-1, b_{j+1,j}) \oplus \ldots \oplus (-1, b_{n,j})$$

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Shift automorphism

Lemma

Let $r, n \in \mathbb{N}$*. Let* $F(r,n)$ *be the Fibonacci group. Then the homomorphism* $\sigma \colon F(r,n) \to F(r,n)$ *defined by*

$$\forall_{0 \leq i \leq n-1}\ \sigma(a_i) = a_{i-1}$$

is an automorphism of $F(r,n)$*.*

Remark

Let's call σ the left shift automorphism of $F(r,n)$.

One dimensional euclidean representations

Theorem

Let $n \in \mathbb{N}$ *be odd. Let* $\Gamma = \langle C_i \mid i = 0, \ldots, n-2 \rangle \subset E(1)$, *where*

$$C_0 = (1, c_0), C_1 = (-1, c_1), \ldots, C_{n-2} = (-1, c_{n-2})$$

and $c_i \in \mathbb{R}$ *for* $i = 0, \ldots, n-2$. *Then there exists an epimorphism*

$$\varphi\colon F(n-1, 2n) \to \Gamma$$

such that

$$\varphi(a_i) = C_i$$

for $i = 0, \ldots, n-2$ *and* $a_0, \ldots, a_{2n-1}$ *are the "cyclic" generators of* $F(n-1, 2n)$.

Every HW-group is cyclic

Theorem

Let $n \in \mathbb{N}$ be odd. Let $\Gamma \subset E(n)$ be a HW-group. Then there exists an epimorphism

$$\Phi\colon F(n-1, 2n) \to \Gamma.$$

Proof

Decomposition of HW-group

- Let
$$[b_{ij}]_{0\leq i,j<n}$$
be a matrix of Γ.
- Let
$$id_\Gamma = \varphi^{(0)} \oplus \ldots \oplus \varphi^{(n-1)}$$
be the euclidean decomposition of id_Γ.
- For every $0 \leq i < n$ there exists epimorphism
$$f_i \colon F(n-1, 2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$$
given by
$$f_i(a_0) = (1, b_{ii}), f_i(a_1) = (-1, b_{i,i+1}), \ldots, f_i(a_{n-1}) = (-1, b_{i,i+n-1}).$$

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$$f_i(a_0) = (1, b_{ii}), f_i(a_1) = (-1, b_{i,i+1}), \ldots, f_i(a_{n-1}) = (-1, b_{i,i+n-1}).$$

Proof

Decomposition of HW-group

- Let
$$\left[b_{ij}\right]_{0\leq i,j<n}$$
be a matrix of Γ.
- Let
$$id_\Gamma = \varphi^{(0)} \oplus \ldots \oplus \varphi^{(n-1)}$$
be the euclidean decomposition of id_Γ.
- For every $0 \leq i < n$ there exists epimorphism
$$f_i \colon F(n-1, 2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$$
given by
$$\forall_{0\leq j<n}\ f_i(a_j) = \left((-1)^{1+\delta_{i,i+j}}, b_{i,i+j}\right)$$

Proof

Left shift automorphism

- $\sigma \in \mathrm{Aut}(F(n-1,2n))$ – left shift automorphism.
- $f_i\sigma^i \colon F(n-1,2n) \to \varphi^{(i)}(\Gamma) \subset E(1)$ for every $0 \leq i < n$.
- $$\forall_{0 \leq i,j < n}\ f_i\sigma^i(a_j) = f_i(a_{j-i}) = \left((-1)^{1+\delta_{i,j}}, b_{i,j}\right)$$

$$\left(\begin{bmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix}, \begin{bmatrix} b_{1,j} \\ \vdots \\ b_{j-1,j} \\ b_{j,j} \\ b_{j+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \right) \qquad \begin{matrix} f_0\sigma^0(a_j) \\ \vdots \\ f_{j-1}\sigma^{j-1}(a_j) \\ f_j\sigma^j(a_j) \\ f_{j+1}\sigma^{j+1}(a_j) \\ \vdots \\ f_n\sigma^n(a_j) \end{matrix}$$

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Proof

The epimorphism

The map

$$\Phi = \bigoplus_{i=0}^{n-1} f_i \sigma^i$$

is the desired epimorphism:

$$\begin{aligned}
\forall_{0 \leq j < n} \; \Phi(a_j) =& \bigoplus_{i=0}^{n-1} f_i \sigma^i (a_j) \\
=& (-1, b_{0,j}) \oplus \ldots \oplus (-1, b_{j-1,j}) \oplus \\
& (1, b_{j,j}) \oplus \\
& (-1, b_{j+1,j}) \oplus \ldots \oplus (-1, b_{n,j}) \\
=& (B_j, b_j).
\end{aligned}$$

Thank you!