Spin structures on almost flat 4-manifolds

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1 Introduction
- Almost flat manifolds
- Spin structures

2 Spin structures
- The algorithm
- Dimension 4

Almost crystallographic groups

- N – connected, simply connected nilpotent Lie group.
- $\mathrm{Aff}(N) := N \rtimes \mathrm{Aut}(N)$ acts on N by

$$(n, \varphi) \cdot m = n\varphi(m),$$

where $n, m \in N, \varphi \in \mathrm{Aut}(N)$.

Definition

- A discrete and cocompact subgroup Γ of $N \rtimes C$ is called an almost crystallographic (AC) group.
- A torsionfree AC-group is called an almost Bieberbach (AB) group.
- Γ – AB-group. N/Γ – almost flat manifold (modeled on N).

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Representations of AC-groups

Theorem (Auslander 1960 – Generalized 1st Bieberbach)

Let $\Gamma \subset N \rtimes C$ be an AC-group. We have the following short exact sequence

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \xrightarrow{\pi} F \longrightarrow 1$$

where $\Lambda = \Gamma \cap N$ is a lattice and F – finite.

Definition

$F \subset C$. Using a monomorphism $C \hookrightarrow \mathrm{O}(n)$ we construct

- holonomy representation $\varphi \colon F \to \mathrm{O}(n)$;
- classyfying representation $\varphi \circ \pi \colon \Gamma \to \mathrm{O}(n)$.

Remark

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Clifford algebra

Definition

Clifford algebra C_n – real associative algebra with 1 generated by $e_1, \dots, e_n$ with the following relations

$$e_i^2 = -1 \text{ and } e_i e_j = -e_j e_i$$

for $1 \leq i < j \leq n$.

Three involutions in C_n

Definition

- ${}^*\colon C_n \to C_n$ defined on the basis of (the vector space) C_n by

$$(e_{i_1} \dots e_{i_k})^* = e_{i_k} \dots e_{i_1}$$

  for $1 \leq i_1 < i_2 < \dots < i_k \leq n$.
- ${}'\colon C_n \to C_n$ defined on the generators of (the algebra) C_n by

$$e_i' = -e_i$$

  for $1 \leq i \leq n$.
- ${}^-\colon C_n \to C_n$ is the composition of the previous involutions

$$\forall_{a \in C_n}\ \overline{a} = (a')^*.$$

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The spin group

Definition

$$\mathrm{Spin}(n) := \{x \in C_n \mid x = x' \text{ and } x\overline{x} = 1\}$$

Example

1. $\mathrm{Spin}(1) = \mathrm{O}(1) = \{\pm 1\}$
2. $\mathrm{Spin}(2) \simeq \mathrm{U}(1)$
3. $\mathrm{Spin}(3) \simeq \mathrm{SU}(2)$
4. $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$
5. $\mathrm{Spin}(5) \simeq \mathrm{Sp}(2)$
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Proposition

We have a short exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(n) \xrightarrow{\lambda_n} \mathrm{SO}(n) \longrightarrow 1$$

where for every $x \in \mathrm{Spin}(n), v \in \mathbb{R}^n$

$$\lambda_n(x)v = xvx^{-1}.$$

Spin structures on almost flat manifolds

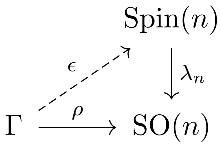
Theorem (Gąsior, Petrosyan, Szczepański 2016)

$\Gamma \subset N \rtimes C$ – *AB-group with classifying representation* $\rho\colon \Gamma \to \mathrm{SO}(n)$.

$$\left\{ \textit{spin structures on } N/\Gamma \right\} \leftrightarrow \left\{ \epsilon\colon \Gamma \to \mathrm{Spin}(n) \mid \lambda_n \epsilon = \rho \right\}$$

$$\begin{array}{ccc} & & \mathrm{Spin}(n) \\ & \overset{\epsilon}{\nearrow} & \downarrow \lambda_n \\ \Gamma & \xrightarrow{\rho} & \mathrm{SO}(n) \end{array}$$

The algorithm



1. Take a finite presentation $\langle S|R\rangle$corresponding to the extension

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow F \longrightarrow 1$$

2. For every $s \in S$ find an element $x_s \in \mathrm{Spin}(n)$ st. $\lambda_n(x_s) = \rho(s)$.

$$\lambda_n^{-1}(\rho(s)) = \{\pm x_s\}$$

3. Check if there is a function $\epsilon' : S \to \mathrm{Spin}(n), s \mapsto \pm x_s$ which preserves the relations of Γ:

$$s_1^{\alpha_1} \dots s_k^{\alpha_k} \in R \;\Rightarrow\; \epsilon'(s_1)^{\alpha_1} \dots \epsilon'(s_k)^{\alpha_k} = 1$$

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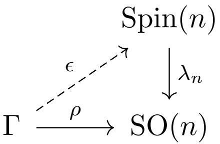

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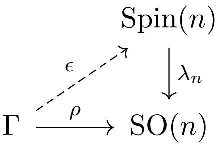

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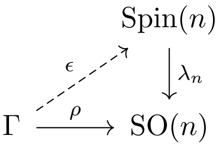

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The algorithm – remarks

Step 3: preserving relations

1. Γ – always finitely presented.

   In `GAP` package `AClib` by B. Eick and K. Dekimpe we have 4-dimensional AC-groups as matrix and polycyclic groups.

2. We can use any extension equivalent to

$$1 \longrightarrow \{\pm 1\} \longrightarrow \lambda_n^{-1}(F) = \widetilde{F} \longrightarrow F \longrightarrow 1.$$

3. Preserving presentation in GAP:

   GroupHomomorphismByImages$(\Gamma, \widetilde{F}, S, \epsilon'(S))$

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We have a central extension

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3. Enough to check when F is a 2-group

Lemma (Gąsior, Petrosyan, Szczepański 2016)

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \xrightarrow{\pi} F \longrightarrow 1$$

Let G be a Sylow 2-subgroup of F. Then

$$N/\Gamma \text{ has a spin structure } \Leftrightarrow N/\pi^{-1}(G) \text{ has a spin structure.}$$

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Lemma

φ is $\mathbb{R}$-equivalent to the representation of the form $F \to \mathrm{GL}(n,\mathbb{Z})$

Lemma (Putrycz, Lutowski 2015)

Every rational representation of a 2-group F is $\mathbb{Q}$-equivalent to the representation of the form $F \to \mathrm{O}(n,\mathbb{Z}) = O(n) \cap \mathrm{GL}(n,\mathbb{Z})$

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Example ($n = 3$)

$$\lambda_3\left(\frac{1 + e_1 e_2}{\sqrt{2}}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Step 1: presentation

$$\begin{array}{ccc} & & \mathrm{Spin}(n) \\ & \overset{\epsilon}{\nearrow} & \big\downarrow \lambda_n \\ \Gamma & \xrightarrow{\ \rho\ } & \mathrm{SO}(n) \end{array}$$

- We have a s.e.s.

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \xrightarrow{\pi} F \longrightarrow 1$$

- $\rho = \varphi\pi$, where $\varphi\colon F \to \mathrm{SO}(n)$ – holonomy representation
- Λ^2 – normal closure of group generated by squares of all elements of Λ
- If ϵ exists then $\Lambda^2 \subset \ker \epsilon$. Isomorphism theorem:

Existence of $\epsilon \Leftrightarrow$ existence of ϵ_2

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Existence of $\epsilon \Leftrightarrow$ existence of ϵ_2

Fact

Up to isomorphism there is a finite number of the groups Γ/Λ^2 in dimension 4.

Some facts

Almost Bieberbach groups in dimension 4

Remark

Classification for AB-groups with abelian Fitting subgroup was made by B. Putrycz and A. Szczepański in 2010.

Based on classification made by Dekimpe in 1996 (`GAP` package `AClib`) for oriented AB-groups with non-abelian Fitting sbgp we have:

1. 9 holonomy groups (up to isomorphism);
2. 9 holonomy representations (up to equivalence) – one for each group;
3. 43 infinite families of AB-groups; in each family
   - every group Γ is defined by a sequence of natural numbers $(k_1, \ldots, k_m)$ and
   - Γ/Λ^2 depends only on $(k_1, \ldots, k_m) \bmod 2$;
4. 127 groups of the form Γ/Λ^2 (up to isomorphism and family);

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Classification for AB-groups with abelian Fitting subgroup was made by B. Putrycz and A. Szczepański in 2010.

Based on classification made by Dekimpe in 1996 (`GAP` package `AClib`) for oriented AB-groups with non-abelian Fitting sbgp we have:

1. 9 holonomy groups (up to isomorphism);
2. 9 holonomy representations (up to equivalence) – one for each group;
3. 43 infinite families of AB-groups; in each family
   - every group Γ is defined by a sequence of natural numbers $(k_1, \ldots, k_m)$ and
   - Γ/Λ^2 depends only on $(k_1, \ldots, k_m) \bmod 2$;
4. 127 groups of the form Γ/Λ^2 (up to isomorphism and family);

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no. of families	15	16	42	44	10
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Example

Family no. 103: presentation

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \xrightarrow{\pi} F \longrightarrow 1$$

$$\Gamma = \langle a,b,c,d,\alpha,\beta \mid \begin{array}{lll} [b,a] = d^{k_1} & [c,a] = 1 & [d,a] = 1 \\ [c,b] = 1 & [d,b] = 1 & [d,c] = 1 \\ \alpha^4 = d^{k_4} & \alpha a = b\alpha d^{k_2} & \alpha b = a^{-1}\alpha d^{k_3} \\ \alpha c = c\alpha & \alpha d = d\alpha & \\ \beta^2 = cd^{k_5} & \beta a = a\beta d^{k_2+k_3} & \beta b = b^{-1}\beta \\ \beta c = c\beta d^{-2k_5} & \beta d = d^{-1}\beta & \alpha\beta = \beta\alpha^3 d^{-k_4} \end{array} \rangle$$

- $\Lambda = \langle a,b,c,d \rangle$
- $F = \langle \overline{\alpha} = \pi(\alpha), \overline{\beta} = \pi(\beta) \rangle$
- (k_1,k_2,k_3,k_4,k_5) for AB-groups:

$$\left.\begin{array}{l} \forall k > 0, k \equiv 0 \bmod 2, (k,0,0,1,0) \\ \forall k > 0, k \equiv 0 \bmod 4, (k,0,0,3,0) \end{array}\right\} \equiv (0,0,0,1,0) \bmod 2$$

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Family no. 103: preimage in $Spin(4)$

$$1 \longrightarrow C_2 \longrightarrow \widetilde{F} \xrightarrow{\lambda_4} F \longrightarrow 1$$

- Holonomy representation $F \to \mathrm{SO}(4,\mathbb{Z})$

$$\overline{\alpha} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \overline{\beta} \mapsto \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\widetilde{F} = \langle A, B, C \rangle$ where $C = -1$ and

$$A = (1 + e_2 e_3)/\sqrt{2}, B = e_1 e_3$$

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- $S = \{a, b, c, d, \alpha, \beta\}$
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check relations of Γ in $\widetilde{F}$.

Out of 32 such functions, 8 preserves the relations.

Corollary

Every almost-flat manifold corresponding to a group in the family no. 103 admits 8 spin structures.

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