Flat Manifolds with Homogeneous Holonomy Representation

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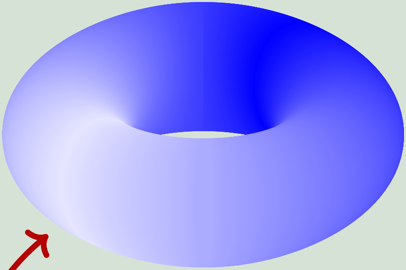
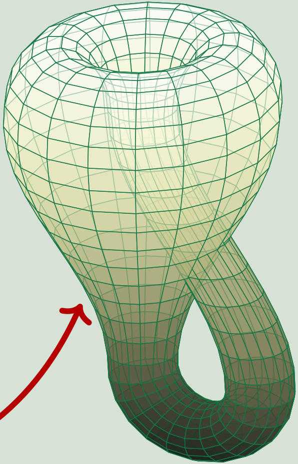
Two flat manifolds

Torus T



Klein bottle K



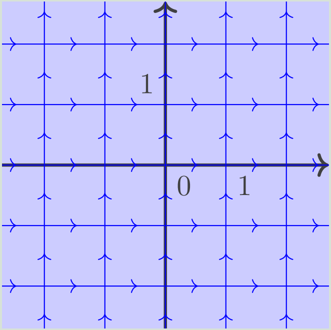
The only 2-dimensional flat manifolds

$\mathrm{A}(n) = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$
$a + A : a \in \mathbb{R}^n, A \in \mathrm{GL}_n(\mathbb{R})$

$(a + A)(b + B)$
$a + Ab + B$

$(a + A)x$
$a + Ax$

$\widetilde{T} = \mathbb{R}^2$



Deck transformations

$$\Gamma_T = \{z + \mathbb{I}_2 : z \in \mathbb{Z}^2\}$$

$\Gamma_T \cong \pi_1(T) \cong \mathbb{Z}^2$

$$T = \mathbb{R}^2/\mathbb{Z}^2$$

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$(a+A)(b+B)$
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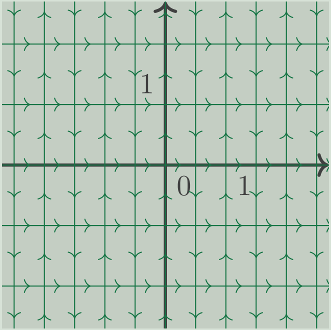
$\widetilde{K} = \mathbb{R}^2$



Deck transformations

$$[\Gamma_K : \Gamma_T] = 2$$

Missing element

$$\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\Gamma_K \cong \pi_1(K)$

$$K = \mathbb{R}^2 / \Gamma_K$$

Γ_T, Γ_K – examples of crystallographic groups

Discrete and cocompact subgroups of $\mathrm{E}(n) := \mathbb{R}^n \rtimes \mathrm{O}(n) \subset \mathrm{A}(n)$.

Theorem (Bieberbach 1911, 1912)

1. *Let $\Gamma \subset \mathrm{E}(n)$ be an n-dimensional crystallographic group. The subgroup $\Gamma \cap (\mathbb{R}^n \times 1)$ of pure translations of Γ is free abelian group of rank n. Moreover it is maximal abelian normal subgroup of Γ of finite index.*
2. *For every $n \in \mathbb{N}$ there is a finite number of isomorphism classes of crystallographic groups of dimension n.*
3. *Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group $\mathrm{A}(n)$.*

$\Gamma \subset E(n)$ – crystallographic group

- Γ fits into a short exact sequence
$$0 \longrightarrow L \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1.$$
- G – finite group – holonomy group of Γ.
- L – faithful G-lattice ($L \cong \mathbb{Z}^n$).
- We get an integral holonomy representation $\varphi\colon G \to \mathrm{GL}(L)$:
$$\forall_{z \in L \subset \Gamma} \forall_{g \in G}\ \varphi_g(z) = \overline{g} z \overline{g}^{-1},$$
where $\pi(\overline{g}) = g$.

Torus

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \Gamma_T \longrightarrow 1 \longrightarrow 1$$

Klein bottle

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \Gamma_K \longrightarrow C_2 \longrightarrow 1$$

Γ–torsionfree crystallographic

$X = \mathbb{R}^n/\Gamma$ – flat manifold

$0 \to L \to \Gamma \to G \to 1$:

$$\alpha \in H^2(G, L)$$

α - special

$\mathrm{res}^G_H\,\alpha \neq 0$ for every $H < G$.

$\mathcal{X} = \{\text{representative } H^G : H < G, |H| - \text{prime}\}$

Theorem (Hiss, Szczepański 1991)

Let $G \neq 1$ be a finite group. If L is an irreducible G-lattice then

$$H^2(G, L)$$

does not contain any special element.

Corollary

1. *The only Bieberbach group with irreducible integral holonomy representation is $\mathbb{Z}$.*
2. *The only flat manifold with $\mathbb{Q}$-irreducible holonomy representation is T^1.*

Special elements in character tables 1

$K = \mathbb{Q}(\xi)$, ξ – primitive G-th root of 1

If L is a G-lattice with character

$$\chi = \chi_1 + \ldots + \chi_k$$

where $\chi_i \in \mathrm{Irr}(G)$ are pairwise non-equal and conjugate in $\mathrm{Gal}(K/\mathbb{Q})$ then L does not contain a special element.

Sketch of the proof

p-special element $\alpha \in H^2(G, L)$

$\operatorname{res}_H \alpha \neq 0$ for every $H \in \mathcal{X}_p = \{H \in \mathcal{X} : |H| = p\}$

$L_p := \mathbb{Z}_p \otimes L$

$i \colon \mathbb{Z} \to \mathbb{Z}_p$

Theorem (Plesken 1989)

α is p-special $\Leftrightarrow$ $i_*\alpha$ is p-special

Sketch of the proof

Lemma (Plesken 1989)

If indecomposable $\mathbb{Z}_p G$-module M is not in the principal $\mathbb{Z}_p G$-block then

$$H^i(G, M) = 0 \textit{ for } i \geq 0.$$

Lemma (Hiss, Szczepański 1991)

Assume L is irreducible and L_p contains an indecomposable direct summand in the principal $\mathbb{Z}_p G$-block. Then every irreducible constituent of $\mathbb{C} \otimes L$ is in the principal p-block of G.

Corollary: $\alpha \in H^2(G, L)$ – p-special

Some constituent of $\mathbb{C} \otimes L$ lies in the principal p-block of G.

EXAMPLE: $G = A_5$

Character table

	$1a$	$2a$	$3a$	$5a$	$5b$
χ_1	1	1	1	1	1
χ_2	3	-1	0	A	$*A$
χ_3	3	-1	0	$*A$	A
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$A = (1-\sqrt{5})/2, \ {*A} = (1+\sqrt{5})/2$

2-blocks

χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	2	1

3-blocks

χ_1	χ_2	χ_3	χ_4	χ_5
1	2	3	1	1

5-blocks

χ_1	χ_2	χ_3	χ_4	χ_5
1	1	1	1	2

Sketch of the proof continued

M – G-lattice with character χ_M

$$\mathrm{Irr}(G,M)=\{\chi\in\mathrm{Irr}\,G : \langle\chi_M,\chi\rangle\neq 0\}$$

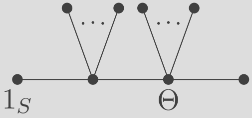
$\mathcal{N}=\{N\triangleleft G : N\text{-minimal}\}$

$$\mathrm{Soc}\,G=\langle\mathcal{N}\rangle=\prod_{N\in\mathcal{N}}N$$

Lemma (Hiss, Szczepański 1991)

L-irreducible, $\alpha\in H^2(G,L)$ – special, $S\triangleleft\mathrm{Soc}\,G$ – simple:

- $\vartheta\in\mathrm{Irr}(G,L)$: ϑ is in the principal p-block for every $p|\#G$
- $\psi\in\mathrm{Irr}(S,L)$: ψ is in the principal p-block for every $p|\#S$
- $\mathrm{Syl}_p(S)$ cyclic: there exists $\Theta\in\mathrm{Irr}(S,L)$ with the following position on the Brauer tree:



Example: $G = A_5$, special elements

L – G-lattice with character χ_L

$$\alpha \in H^2(G, L) \text{ – special}$$

Character table

	$1a$	$2a$	$3a$	$5a$	$5b$
χ_1	1	1	1	1	1
χ_2	3	-1	0	A	$^*A$
χ_3	3	-1	0	$^*A$	A
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$A = (1-\sqrt{5})/2,\ {}^*A = (1+\sqrt{5})/2$

Brauer tree, $p = 5$

$\langle \chi_L, \chi_2 \rangle, \langle \chi_L, \chi_3 \rangle > 0$

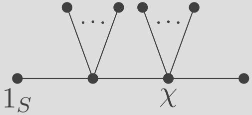
Brauer tree, $p = 3$

$\langle \chi_L, \chi_4 \rangle > 0$

Sketch of the proof: (almost) the end

$\mathcal{S}(G) \subset \operatorname{Irr} G$. $\chi \in \mathcal{S}(G)$ if for every $p | \# G$:

- χ is in the principal p-block
- If $\operatorname{Syl}_p(G)$ is cyclic: χ has the following position on the Brauer tree:



Proposition (Hiss, Szczepański 1991)

If S is a finite nonabelian simple group, then $\mathcal{S}(S) = \emptyset$.

Homogeneous representations

Homogeneous G-lattice L:

$$\mathbb{Q} \otimes L = V \oplus \ldots \oplus V$$

L – homogeneous:

$$\operatorname{Irr}(G, L) = \operatorname{Irr}(G, \mathbb{Q} \otimes L) = \operatorname{Irr}(G, V)$$

Theorem

Let $G \neq 1$ be a finite group. If L is a homogeneous G-lattice then

$$H^2(G, L)$$

does not contain any special element.

Corollary

1. *The only Bieberbach groups with homogeneous integral holonomy representation are f.g. free abelian groups.*
2. *The only flat manifolds with $\mathbb{Q}$-homogeneous holonomy representation are flat tori.*

Special elements in character tables 2

$K = \mathbb{Q}(\xi)$, ξ – primitive G-th root of 1

If L is a G-lattice with character

$$\chi = \chi_1 + \ldots + \chi_k$$

where $\chi_i \in \mathrm{Irr}(G)$ are pairwise ~~non-equal and~~ conjugate in $\mathrm{Gal}(K/\mathbb{Q})$ then L does not contain a special element.

Flat Kähler manifolds

Γ – torsionfree discrete and cocompact sgp of $\mathbb{C}^n \rtimes U(n)$

$\mathbb{C}^n/\Gamma$ – flat Kähler manifold

$\pi\colon \mathbb{C}^n \rtimes U(n) \to U(n)$ – projection

$G = \pi(\Gamma)$ – (finite) holonomy gp of Γ

$\varphi = \mathrm{id}_G$

χ – character of φ

$\chi+\overline{\chi}$ – character of integral hol rep of Bieberbach gp

χ – $\mathbb{C}$-homogeneous $\Rightarrow \chi + \overline{\chi}$ – $\mathbb{Q}$-homogeneous

Theorem

Holonomy representation of a Kähler flat manifold, which is not a flat torus, contains at least two $\mathbb{C}$*-homogeneous components. In particular it is reducible.*

Thank you!