Flat Manifolds with Holonomy Representation of Quaternionic Type

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Flat manifolds

Crystallographic groups

Affine motions in $\mathbb{R}^n$

$$A(n) := \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$$

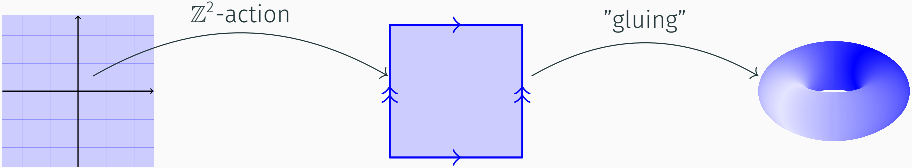
Isometries of $\mathbb{R}^n$

$$E(n) := \mathbb{R}^n \rtimes O(n)$$

Crystallographic group

Discrete and co-compact subgroup of $E(n)$.

Orbit spaces and Bieberbach groups



When orbit spaces are manifolds?

When crystallographic groups are torsion-free – Bieberbach groups.

Constructing Bieberbach groups

Structure of crystallographic groups (Bieberbach 1911)

Γ – crystallographic. Γ fits into a short exact sequence

$$0 \longrightarrow L \longrightarrow \Gamma \longrightarrow G \longrightarrow 1. \tag{1}$$

G – finite group – holonomy group of Γ.
L – faithful G-lattice ($L \cong \mathbb{Z}^n$).

When crystallographic is Bieberbach?

Let $\alpha \in H^2(G, L)$ correspond to (1). Γ is Bieberbach iff α is special:

$$\mathrm{res}^G_C\, \alpha \neq 0$$

for all cyclic $C < G$ of prime order.

What defines a Bieberbach group?

Faithful G-lattice L with special element $\alpha \in H^2(G, L)$.

Problem

Types of real modules

G – finite group, V – $\mathbb{R}G$-module

Decomposition into irreducible components:

$$V = V_1 \oplus \ldots \oplus V_k$$

For every irreducible component V_i we have

$$\mathrm{End}_{\mathbb{R}G}(V_i) = \left\{ \begin{array}{ccccc} \mathbb{R} & : & \mathbb{C} \otimes_{\mathbb{R}} V_i = U & : & 1 \\ \mathbb{C} & : & \mathbb{C} \otimes_{\mathbb{R}} V_i = U \oplus \overline{U} & : & 0 \\ \mathbb{H} & : & \mathbb{C} \otimes_{\mathbb{R}} V_i = U \oplus U & : & -1 \end{array} \right\} = \nu_2(\chi_U) \Leftrightarrow \chi_U \in \left\{ \begin{array}{l} \mathrm{Irr}_{\mathbb{R}}(G) \\ \mathrm{Irr}_{\mathbb{C}}(G) \\ \mathrm{Irr}_{\mathbb{H}}(G) \end{array} \right.$$

χ_U – character of irreducible $\mathbb{C}G$-module U, $U \not\cong \overline{U}$

$\nu_2(\chi) = \sum_{g \in G} \chi(g^2)$ – Frobenius-Schur indicator

We get (unique) decomposition

$$V = V_{\mathbb{R}} \oplus V_{\mathbb{C}} \oplus V_{\mathbb{H}}$$

Problem

Recall

Bieberbach group Γ is defined by faithful G-lattice L and special element $\alpha \in H^2(G, L)$.

Question

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Can we find a Bieberbach group Γ st. $\mathbb{R} \otimes_{\mathbb{Z}} L = (\mathbb{R} \otimes_{\mathbb{Z}} L)_{\mathbb{F}}$?

For complex and quaternionic case:

We would get kähler ($G \subset \mathrm{U}(n)$) and hyperkähler ($G \subset \mathrm{Sp}(n)$) structure in a non-trivial way – not coming from inclusion $\mathrm{Sp}(n) \subset \mathrm{U}(2n) \subset \mathrm{O}(4n)$.

In real and complex case the answer is yes:

(1) 3-dimensional with $G = C_2^2$ (Hantzsche-Wendt 1935);

(2) 8-dimensional with $G = C_3^2$ and $L^G = 0$ (Hiller-Sah 1986).

Restrictions on holonomy group

Γ – Bieberbach group of quaternionic type defined by G-lattice L and $\alpha \in H^2(G, L)$:

1. $|G|$ is even, otherwise $g \mapsto g^2$ is bijection and for $\chi \in \mathrm{Irr}(G)$:
$$\nu_2(\chi) = \sum \chi(g^2) = \sum \chi(g) = \langle \chi, 1 \rangle \in \{0, 1\}.$$
2. G is non-abelian, otherwise $\nu_2(\chi) \in \{0, 1\}$ for $\chi \in \mathrm{Irr}(G)$.
3. $Z(G)$ is elementary abelian 2-group, otherwise:
   - $z \in Z(G)$ – of order 4 or p (odd prime).
   - $\chi \in \mathrm{Irr}(G)$ – summand of χ_L st. $z^2 \notin \ker \chi$.
   - Schur's lemma: $\mathrm{res}_{Z(G)}\, \chi = \chi(1)\lambda$ for some $\lambda \in \mathrm{Irr}(Z(G))$.

   Hence $\chi(z) \in \mathbb{C} \setminus \mathbb{R}$ and $\nu_2(\chi) = 0$.
4. No cyclic Sylow subgroup of G has normal complement:
   (Han-Sah 1986): implied by $L^G = 0$.
5. 2-Sylow subgroup of G is not cyclic:
   Cayley normal 2-complement theorem (1878).

Restrictions on holonomy group

Γ – Bieberbach group of quaternionic type defined by G-lattice L and $\alpha \in H^2(G, L)$:

6. Let $\mathrm{I}(G) := |\{g \in G : g^2 = 1\}|$: $\mathrm{I}(G) \leq |G|/2$ and $\mathrm{I}(G) < \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)$:
   1st (Wall 1970): otherwise $\mathrm{Irr}(G) = \mathrm{Irr}_{\mathbb{R}}(G)$.
   2nd (Frobenius-Schur formula):
   $$\mathrm{I}(G) = \sum_{\chi \in \mathrm{Irr}(G)} \nu_2(\chi)\chi(1) = \sum_{\chi \in \mathrm{Irr}_{\mathbb{R}}(G)} \chi(1) - \sum_{\chi \in \mathrm{Irr}_{\mathbb{H}}(G)} \chi(1).$$
7. $|\mathrm{Irr}_{\mathbb{H}}(G)| > 1$:
   (L. 2018): $\mathbb{C} \otimes_{\mathbb{Z}} L$ contains at least two non-isomorphic components.
8. $\forall_{z \in Z(G) \setminus \{1\}} \exists_{\chi, \psi \in \mathrm{Irr}_{\mathbb{H}}(G)}$ $\chi(z) = \chi(1)$ and $\psi(z) = -\psi(1)$:
   Otherwise L not faithful or α not special.

Only one group of order ≤ 64 satisfies the above conditions.

Example

gap> G := SmallGroup(64,245);

$G = \langle a, b, c, d \rangle$ fits into central extension

$$1 \longrightarrow C_2^2 \longrightarrow G \longrightarrow C_2^4 \longrightarrow 1.$$

$a^2 = c^2, b^2 = d^2$ generate $Z(G)$ and

$$\begin{array}{lll} [a,b] = a^2 & [a,c] = a^2b^2 & [a,d] = b^2 \\ & [b,c] = a^2 & [b,d] = a^2b^2 \\ & & [c,d] = 1 \end{array}$$

3 characters with FS-indicator -1:

	1	a^2	b^2	a^2b^2	$G \setminus Z(G)$
χ_1	4	4	-4	-4	0
χ_2	4	-4	4	-4	0
χ_3	4	-4	-4	4	0

Characters conjugate

$\chi_i = \chi_1 f_i$ for some $f_i \in \mathrm{Aut}(G)$

$Z_i := \ker \chi_i$

$$f_i(Z_i) = Z_1$$

Idea for module with special element

For G-lattice L and $f \in \mathrm{Aut}(G)$ we have

1. G-lattice $(L^f, \cdot_f)$: $L^f = L, g \cdot_f l = f(g)l$
2. Commutative diagram for $H < G$, where $(f_{|H})^*$ – isomorphism:

$$\begin{array}{ccc} H^2(G, L) & \xrightarrow{f^*} & H^2(G, L^f) \\ \downarrow{\scriptstyle \mathrm{res}_H} & & \downarrow{\scriptstyle \mathrm{res}_{f(H)}} \\ H^2(H, \mathrm{res}_H L) & \xrightarrow{(f_{|H})^*} & H^2(f(H), \mathrm{res}_{f(H)} L^f) \end{array}$$

Corollary

If we find a G-lattice L and $\alpha \in H^2(G, L)$ st. $\mathrm{res}_{Z_1} \alpha \neq 0$ then

$$\mathrm{res}_{Z_i} f_i^*(\alpha) = (f_{|Z_1})^* \,\mathrm{res}_{Z_1} \alpha \neq 0$$

and $\alpha + f_2^*(\alpha) + f_3^*(\alpha) \in H^2(G, L \oplus L^{f_2} \oplus L^{f_3})$ is special.

The lattice: first attempt

Some GAP code

```
gap> rep := IrreducibleRepresentations(G)[...]; # chi_1
gap> FieldOfMatrixGroup( Image(rep) );
GaussianRationals                               # expected
                                                # (Schur index)
```

Remarks

1. Smallest lattice dimension to work with: 8.
2. Easy computation: $H^2(G, L)$. But:
   For every L with $\chi_L = 2\chi_1$ we've tried we got $\mathrm{res}_{C_1} \alpha = 0$ for all $\alpha \in H^2(G, L)$.
3. Hard computation: determine all lattices with character $2\chi_1$.
   It would take too long to wait for...

The lattice: successful attempt

$L' := \operatorname{ind}_{C_1}^{G} \mathbb{Z}$. By Shapiro's lemma $H^2(G, L') = H^2(C_1, \mathbb{Z}) = \mathbb{Z}/2$ and

$$\operatorname{res}_{C_1} \alpha' \neq 0 \text{ for } 0 \neq \alpha' \in H^2(G, L')$$

Quaternionic components

$$\langle \chi_{L'}, \chi_i \rangle = \begin{cases} 4, & i = 1 \\ 0, & i \neq 1 \end{cases}$$

Basis for L

$$B = \frac{2\chi_1(1)}{|G|} \sum_{g \in G} \overline{2\chi_1(g)} \rho_{L'}(g)$$

We get "quaternionic" Bieberbach group:

$\chi_L = 4\chi_1$ and for $0 \neq \alpha \in H^2(G, L) = \mathbb{Z}/2$

$$\operatorname{res}_{C_1} \alpha \neq 0.$$

Lemma

Let G be a finite group and p a prime number. Then $O_{p'}(G)$ is contained in the kernel of every $\chi \in \mathrm{Irr}(G)$ in the principal p-block.

Lemma (Hiss, Szczepański 1991)

Let G be a finite group and L be a G-lattice. If $H^2(G, L)$ contains a special element then for every prime divisor p of $|G|$ there exists a constituent of $\mathbb{C} \otimes_{\mathbb{Z}} L$ which lies in the principal p-block of G.

Notes on holonomy groups

Theorem

Let Γ be quaternionic Bieberbach group with holonomy group G. Then G is not:

(i) $\mathrm{SL}_2(\mathbb{F}_q), \mathrm{PSL}_2(\mathbb{F}_q)$, *where q is a power of a prime;* *(char. table + 1st lemma)*

(ii) $A_n, 2.A_n, S_n, 2.S_n, n \geq 5$; *(Clifford theorem)*

(iii) *a perfect central extension of a sporadic simple group.* *(Atlas+both lemmas)*

Theorem (Willems 1977)

If a finite group G is non-abelian and all its non-linear characters have Frobenius-Schur indicator equal to -1 then G is a 2-group.

Thank you!