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Complex Vasquez invariant

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Vasquez number

- 1 Vasquez number
- 2 Complex structures
- 3 Complex Vasquez invariant
- 4 Complex geometry

Global assumption

- ▶ G – a finite group
- ▶ \mathcal{X} – set of representatives of $\{H^G : H < G \text{ and } |H| \text{ is a prime}\}$
- ▶ all modules – finite-dimensional (over \mathbb{C})

Definition

Let L be a G -lattice. An element $\alpha \in H^2(G, L)$ is **special**, if

$$\text{res}_H^G \alpha \neq 0$$

for every $H \in \mathcal{X}$.

Theorem (Vasquez 1970)

There exists a number $n(G)$ such that if L is a G -lattice with a special element α then there exists a \mathbb{Z} -pure submodule L' of L st.

- 1 $\text{rk}_{\mathbb{Z}}(L/L') = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} L/L') = \dim_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} L/L') \leq n(G)$,
- 2 $\nu_*(\alpha)$ is special,

where $\nu: L \rightarrow L/L'$ is the natural homomorphism.

Proof.

- 1 For $H \in \mathcal{X}$:

- ▶ $\text{res}_H L = L_0 \oplus L'_0$, $L_0 \cong \mathbb{Z}$ st. $\text{res}_H \alpha = \alpha_0 + \alpha'_0$ and $\alpha_0 \neq 0$
- ▶ $L'_H := \cap_{g \in G} gL'_0$. Note $\text{rk}_{\mathbb{Z}}(L/L'_H) \leq \text{rk}_{\mathbb{Z}} \text{ind}_H^G L_0 = [G:H]$.

- 2 $L' := \cap_{H \in \mathcal{X}} L'_H$. By construction $\nu_*(\alpha)$ is special and

$$\text{rk}_{\mathbb{Z}}(L/L') \leq \sum_{H \in \mathcal{X}} \text{rk}_{\mathbb{Z}}(L/L'_H) \leq \sum_{H \in \mathcal{X}} [G:H].$$

Hence $n(G) \leq \sum [G:H]$ (Cliff, Weiss 1990). □

Complex structures

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Definition

Let V be a KG -module, where K is \mathbb{Z} , \mathbb{Q} or \mathbb{R} .

- ▶ **Complex structure** on V : $J \in \text{End}_{\mathbb{R}G}(\mathbb{R} \otimes_K V)$ st. $J^2 = -id$.
- ▶ A module admitting a complex structure is called **essentially complex**.

Corollary

$\mathbb{R} \otimes_K V$ is complex vector space with $i \cdot v := Jv$ and

$$\dim_{\mathbb{R}}(\mathbb{R} \otimes_K V) = 2 \dim_{\mathbb{C}}(\mathbb{R} \otimes_K V).$$

Theorem (Johnson 1990)

Let V be an $\mathbb{R}G$ -module. The following are equivalent:

- 1 V is essentially complex.
- 2 Every homogeneous component of V is essentially complex.
- 3 Every absolutely irreducible component of V occurs with even multiplicity.

Frobenius-Schur indicator

Lemma

Let V be a simple $\mathbb{Q}G$ -module with character χ_V . Let $\chi_s \in \text{Irr}(G)$ be such that $(\chi_V, \chi_s) \neq 0$. Then

$$(\chi_V, \chi) \neq 0 \Rightarrow \nu_2(\chi) = \nu_2(\chi_s)$$

for every $\chi \in \text{Irr}(G)$.

Definition

We say that V is of type

- ▶ \mathbb{R} if $\nu_2(\chi_s) = 1$,
- ▶ \mathbb{C} if $\nu_2(\chi_s) = 0$,
- ▶ \mathbb{H} if $\nu_2(\chi_s) = -1$.

Remark

The above definition applies to simple G -lattices and $\mathbb{R}G$ -modules. In the latter case we get that V is of type K iff $\text{End}_{\mathbb{R}G}(V) \cong K$.

Definition

Let V be an irreducible $\mathbb{Q}G$ -module with character χ_V . Let $\chi_s \in \text{Irr}(G)$ with $(\chi_V, \chi_s) \neq 0$. Define

$$m(V) := m_{\mathbb{Q}}(\chi_s).$$

For an irreducible G -lattice L we define $m(L) := m(\mathbb{Q} \otimes_{\mathbb{Z}} L)$.

Proposition

Let L be an irreducible G -lattice. The following are equivalent:

- 1 L is essentially complex.
- 2 $\mathbb{Q} \otimes_{\mathbb{Z}} L$ is essentially complex.
- 3 L is of type \mathbb{C}, \mathbb{H} or $m(L)$ is even.

Theorem (Johnson 1990)

A G -lattice L is essentially complex iff every simple component V of $\mathbb{Q} \otimes_{\mathbb{Z}} L$ of type \mathbb{R} with odd $m(V)$ occurs with even multiplicity.

Complex Vasquez invariant

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Theorem (Gašior, L., Szczepański 2022)

There exists a number $n_{\mathbb{C}}(G)$ such that if L is an essentially complex G -lattice with a special element α then there exists a \mathbb{Z} -pure submodule L' of L st.

- 1 L/L' is essentially complex,
- 2 $\text{rk}_{\mathbb{Z}}(L/L')/2 = \dim_{\mathbb{C}}(\mathbb{R} \otimes_{\mathbb{Z}} L/L') \leq n_{\mathbb{C}}(G)$,
- 3 $\nu_*(\alpha)$ is special,

where $\nu: L \rightarrow L/L'$ is the natural homomorphism.

Proof.

- 1 L'' – \mathbb{Z} -pure submodule of L from Vasquez construction.
- 2 $\mathbb{Q} \otimes_{\mathbb{Z}} L'' = V' \oplus \sum_i n_i V_i$ – sum over all simple V_i with odd $m(V_i)$ and n_i .
- 3 $L' := L'' \cap (V' \oplus \sum_i (n_i - 1)V_i)$ – ess. complex $\Rightarrow L/L'$ – ess. complex.
- 4 $\text{rk}_{\mathbb{Z}}(L/L') \leq 2 \text{rk}_{\mathbb{Z}}(L/L'') \Rightarrow n_{\mathbb{C}}(G) \leq n(G)$.
- 5 $L \xrightarrow{\nu} L/L' \xrightarrow{\pi} L/L'': (\pi\nu)_*(\alpha)$ special $\Rightarrow \nu_*(\alpha)$ special.



Complex Vasquez invariant

Theorem (Gašior, L., Szczepański 2022)

There exists a number $n_{\mathbb{C}}(G)$ such that if L is an essentially complex G -lattice with a special element α then there exists a \mathbb{Z} -pure submodule L' of L st.

- 1 L/L' is essentially complex,
- 2 $\text{rk}_{\mathbb{Z}}(L/L')/2 = \dim_{\mathbb{C}}(\mathbb{R} \otimes_{\mathbb{Z}} L/L') \leq n_{\mathbb{C}}(G)$,
- 3 $\nu_*(\alpha)$ is special,

where $\nu: L \rightarrow L/L'$ is the natural homomorphism.

Proposition

$$n(G)/2 \leq n_{\mathbb{C}}(G) \leq n(G)$$

Example

$$5 \leq n_{\mathbb{C}}(C_2^2) \leq 6 = n(C_2^2)$$

and

$$n_{\mathbb{C}}(C_3^2) = n(C_3^2)/2 = 6.$$

Theorem (Gašior, L., Szczepański 2022)

There exists a number $n_{\mathbb{C}}(G)$ such that if L is an essentially complex G -lattice with a special element α then there exists a \mathbb{Z} -pure submodule L' of L st.

- 1 L/L' is essentially complex,
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where $\nu: L \rightarrow L/L'$ is the natural homomorphism.

Proposition

$$n_{\mathbb{C}}(G) \leq \frac{n(G)}{2} + \frac{1}{2} \sum m_{\mathbb{Q}}(\chi) \chi(1),$$

where the sum is taken over $\chi \in \text{Irr}(G)$ with $\nu_2(\chi) = 1$.

Complex geometry

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Definition

A homomorphism f of $\mathbb{R}G$ -modules V and W , with complex structures J_V and J_W , is **holomorphic** if it is \mathbb{C} -linear, i.e.

$$fJ_V = J_Wf.$$

Lemma

Keeping the above notation, assume that f is holomorphic.

- 1 Kernel of f is J_V -invariant.
- 2 If f is onto, then J_W is uniquely determined by J_V .

Corollary

Let $f: V \rightarrow W$ be an epimorphism of $\mathbb{R}G$ -modules, with complex structures J_V and J_W . Then f is holomorphic iff $\ker f$ is J_V -invariant.

Lemma

Let K be \mathbb{R}, \mathbb{C} or \mathbb{H} and $n \in \mathbb{N}$. Let $J \in \text{GL}_n(K)$ be such that $J^2 = -I$. Then there exists $A \in \text{GL}_n(K)$ st. J^A is in canonical form, i.e.

1 if $K = \mathbb{R}$

$$J^A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

2 if $K = \mathbb{C}$

$$J^A = \text{diag}(i, \dots, i, -i, \dots, -i)$$

3 if $K = \mathbb{H}$ (Wiegmann, 1954)

$$J^A = \text{diag}(i, \dots, i)$$

Lemma

Let K be \mathbb{R}, \mathbb{C} or \mathbb{H} , $n \in \mathbb{N}$ and $A \in \text{GL}_n(K)$. Then A may be continuously transformed to a diagonal form with the usage of "elementary non-negative transformations", for example:

$$\begin{bmatrix} 1 & t\alpha \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} (1-t) + t\beta & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \cos(t\pi/2) & -\sin(t\pi/2) \\ \sin(t\pi/2) & \cos(t\pi/2) \end{bmatrix},$$

where $\alpha, \beta \in K, \beta \notin \mathbb{R}_-, t \in [0, 1]$.

$$\mathfrak{D}(G) := \{J \in \text{End}_{\mathbb{R}G}(\mathbb{R} \otimes_K L) : J^2 = -id\}$$

Proposition (Gašior, L., Szczepański 2022)

Let $V' \subset V$ be essentially complex $\mathbb{R}G$ -modules and let J be a complex structure on V . Then J can be continuously deformed in $\mathfrak{D}(G)$ to J' such that V' is J' -invariant.

Proof.

- 1 All elements of $\text{End}_{\mathbb{R}G}(V)$ preserve homogeneous components of V .
- 2 U – simple in homogeneous component W :

$$W = \sum_{i=1}^m U \oplus \sum_{i=m+1}^n U \subset V' \oplus V'' = V$$

- 3 U is of type $K = \text{End}_{\mathbb{R}G}(U)$: $\text{End}_{\mathbb{R}G}(W) = M_n(K)$.
- 4 $K = \mathbb{R} \Rightarrow m$ and n are even.
- 5 Lemmas on the previous slide give us desired deformation on W , and hence on V .





Thank you!