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Complex Vasquez invariant

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Vasquez number

Special element

Global assumption

- G – a finite group
- $\mathscr{X}$ – set of representatives of $\{H^G : H < G \text{ and } |H| \text{ is a prime}\}$
- all modules – finite-dimensional (over $\mathbb{C}$)

Definition

Let L be a G-lattice. An element $\alpha \in H^2(G, L)$ is special, if

$$\operatorname{res}^G_H \alpha \neq 0$$

for every $H \in \mathscr{X}$.

Vasquez number

Theorem (Vasquez 1970)

There exists a number $n(G)$ such that if L is a G-lattice with a special element α then there exists a $\mathbb{Z}$-pure submodule L' of L st.

1. $\mathrm{rk}_{\mathbb{Z}}(L/L') = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} L/L') = \dim_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} L/L') \leq n(G)$,
2. $\nu_*(\alpha)$ *is special,*

where $\nu \colon L \to L/L'$ is the natural homomorphism.

Proof.

1. For $H \in \mathscr{X}$:
   - $\mathrm{res}_H L = L_0 \oplus L_0'$, $L_0 \cong \mathbb{Z}$ st. $\mathrm{res}_H \alpha = \alpha_0 + \alpha_0'$ and $\alpha_0 \neq 0$
   - $L_H' := \cap_{g \in G} g L_0'$. Note $\mathrm{rk}_{\mathbb{Z}}(L/L_H') \leq \mathrm{rk}_{\mathbb{Z}} \mathrm{ind}_H^G L_0 = [G : H]$.
2. $L' := \cap_{H \in \mathscr{X}} L_H'$. By construction $\nu_*(\alpha)$ is special and

$$\mathrm{rk}_{\mathbb{Z}}(L/L') \leq \sum_{H \in \mathscr{X}} \mathrm{rk}_{\mathbb{Z}}(L/L_H') \leq \sum_{H \in \mathscr{X}} [G : H].$$

Hence $n(G) \leq \sum [G : H]$ (Cliff, Weiss 1990). $\square$

Complex structures

Complex structures

Definition

Let V be a KG-module, where K is $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$.

- Complex structure on V: $J \in \operatorname{End}_{\mathbb{R}G}(\mathbb{R} \otimes_K V)$ st. $J^2 = -id$.
- A module admitting a complex structure is called essentially complex.

Corollary

$\mathbb{R} \otimes_K V$ is complex vector space with $i \cdot v := Jv$ and

$$\dim_{\mathbb{R}}(\mathbb{R} \otimes_K V) = 2 \dim_{\mathbb{C}}(\mathbb{R} \otimes_K V).$$

Theorem (Johnson 1990)

Let V be an $\mathbb{R}G$-module. The following are equivalent:

1. *V is essentially complex.*
2. *Every homogeneous component of V is essentially complex.*
3. *Every absolutely irreducible component of V occurs with even multiplicity.*

Frobenius-Schur indicator

Lemma

Let V be a simple $\mathbb{Q}G$-module with character χ_V. Let $\chi_s \in \mathrm{Irr}(G)$ be such that $(\chi_V, \chi_s) \neq 0$. Then

$$(\chi_V, \chi) \neq 0 \Rightarrow \nu_2(\chi) = \nu_2(\chi_s)$$

for every $\chi \in \mathrm{Irr}(G)$.

Definition

We say that V is of type

- $\mathbb{R}$ if $\nu_2(\chi_s) = 1$,
- $\mathbb{C}$ if $\nu_2(\chi_s) = 0$,
- $\mathbb{H}$ if $\nu_2(\chi_s) = -1$.

Remark

The above definition applies to simple G-lattices and $\mathbb{R}G$-modules. In the latter case we get that V is of type K iff $\mathrm{End}_{\mathbb{R}G}(V) \cong K$.

Complex structures

Definition

Let V be an irreducible $\mathbb{Q}G$-module with character χ_V. Let $\chi_s \in \operatorname{Irr}(G)$ with $(\chi_V, \chi_s) \neq 0$. Define

$$m(V) := m_{\mathbb{Q}}(\chi_s).$$

For an irreducible G-lattice L we define $m(L) := m(\mathbb{Q} \otimes_{\mathbb{Z}} L)$.

Proposition

Let L be an irreducible G-lattice. The following are equivalent:

1. *L is essentially complex.*
2. *$\mathbb{Q} \otimes_{\mathbb{Z}} L$ is essentially complex.*
3. *L is of type $\mathbb{C}, \mathbb{H}$ or $m(L)$ is even.*

Theorem (Johnson 1990)

A G-lattice L is essentially complex iff every simple component V of $\mathbb{Q} \otimes_{\mathbb{Z}} L$ of type $\mathbb{R}$ with odd $m(V)$ occurs with even multiplicity.

Complex Vasquez invariant

Complex Vasquez invariant

Theorem (Gąsior, L., Szczepański 2022)

There exists a number $n_{\mathbb{C}}(G)$ such that if L is an essentially complex G-lattice with a special element α then there exists a $\mathbb{Z}$-pure submodule L' of L st.

1. *L/L' is essentially complex,*
2. $\mathrm{rk}_{\mathbb{Z}}(L/L')/2 = \dim_{\mathbb{C}}(\mathbb{R} \otimes_{\mathbb{Z}} L/L') \leq n_{\mathbb{C}}(G)$,
3. *$\nu_*(\alpha)$ is special,*

where $\nu\colon L \to L/L'$ is the natural homomorphism.

Proof.

1. L'' – $\mathbb{Z}$-pure submodule of L from Vasquez construction.
2. $\mathbb{Q} \otimes_{\mathbb{Z}} L'' = V' \oplus \sum_i n_i V_i$ – sum over all simple V_i with odd $m(V_i)$ and n_i.
3. $L' := L'' \cap \left(V' \oplus \sum_i (n_i - 1) V_i\right)$ – ess. complex $\Rightarrow L/L'$ – ess. complex.
4. $\mathrm{rk}_{\mathbb{Z}}(L/L') \leq 2\,\mathrm{rk}_{\mathbb{Z}}(L/L'') \Rightarrow n_{\mathbb{C}}(G) \leq n(G)$.
5. $L \xrightarrow{\nu} L/L' \xrightarrow{\pi} L/L''$: $(\pi\nu)_*(\alpha)$ special $\Rightarrow \nu_*(\alpha)$ special.

□

Complex Vasquez invariant

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where $\nu\colon L \to L/L'$ is the natural homomorphism.

Proposition

$$n(G)/2 \leq n_{\mathbb{C}}(G) \leq n(G)$$

Example

$$5 \leq n_{\mathbb{C}}(C_2^2) \leq 6 = n(C_2^2)$$

and

$$n_{\mathbb{C}}(C_3^2) = n(C_3^2)/2 = 6.$$

Complex Vasquez invariant

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Proposition

$$n_{\mathbb{C}}(G) \leq \frac{n(G)}{2} + \frac{1}{2} \sum m_{\mathbb{Q}}(\chi)\chi(1),$$

where the sum is taken over $\chi \in \mathrm{Irr}(G)$ with $\nu_2(\chi) = 1$.

Complex geometry

Holomorphic map

Definition

A homomorphism f of $\mathbb{R}G$-modules V and W, with complex structures J_V and J_W, is holomorphic if it is $\mathbb{C}$-linear, i.e.

$$fJ_V = J_W f.$$

Lemma

Keeping the above notation, assume that f is holomorphic.

1. *Kernel of f is J_V-invariant.*
2. *If f is onto, then J_W is uniquely determined by J_V.*

Corollary

Let $f\colon V \to W$ be an epimorphism of $\mathbb{R}G$-modules, with complex structures J_V and J_W. Then f is holomorphic iff $\ker f$ is J_V-invariant.

Canonical forms of J

Lemma

Let K be $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $n \in \mathbb{N}$. Let $J \in \mathrm{GL}_n(K)$ be such that $J^2 = -I$. Then there exists $A \in \mathrm{GL}_n(K)$ st. J^A is in canonical form, i.e.

1. *if $K = \mathbb{R}$*

$$J^A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

2. *if $K = \mathbb{C}$*

$$J^A = \mathrm{diag}(i, \ldots, i, -i, \ldots, -i)$$

3. *if $K = \mathbb{H}$ (Wiegmann, 1954)*

$$J^A = \mathrm{diag}(i, \ldots, i)$$

Lemma

Let K be $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, $n \in \mathbb{N}$ and $A \in \mathrm{GL}_n(K)$. Then A may be continuously transformed to a diagonal form with the usage of "elementary non-negative transformations", for example:

$$\begin{bmatrix} 1 & t\alpha \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} (1-t) + t\beta & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \cos(t\pi/2) & -\sin(t\pi/2) \\ \sin(t\pi/2) & \cos(t\pi/2) \end{bmatrix},$$

where $\alpha, \beta \in K, \beta \notin \mathbb{R}_-, t \in [0,1]$.

Deformations

$$\mathfrak{D}(G) := \{J \in \mathrm{End}_{\mathbb{R}G}(\mathbb{R} \otimes_K L) : J^2 = -id\}$$

Proposition (Gąsior, L., Szczepański 2022)

Let $V' \subset V$ be essentially complex $\mathbb{R}G$-modules and let J be a complex structure on V. Then J can be continuously deformed in $\mathfrak{D}(G)$ to J' such that V' is J'-invariant.

Proof.

1. All elements of $\mathrm{End}_{\mathbb{R}G}(V)$ preserve homogeneous components of V.
2. U – simple in homogeneous component W:
$$W = \sum_{i=1}^{m} U \oplus \sum_{i=m+1}^{n} U \subset V' \oplus V'' = V$$
3. U is of type $K = \mathrm{End}_{\mathbb{R}G}(U)$: $\mathrm{End}_{\mathbb{R}G}(W) = M_n(K)$.
4. $K = \mathbb{R} \Rightarrow m$ and n are even.
5. Lemmas on the previous slide give us desired deformation on W, and hence on V.

□

Thank you!