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Minimal non-solvable Bieberbach groups

with Andrzej Szczepański

Main question

Question (Hillman 2022)

What is a minimal Hirsch length $h(\Gamma)$ of a torsion-free virtually polycyclic non-solvable group Γ?

Three cases

Theorem (Hillman 2023)

1. Γ – *virtually solvable (general case):*
$$h(\Gamma) \geq 10.$$
2. Γ – *virtually nilpotent and* $h(\Gamma) \leq 14$, *then the Fitting subgroup is of nilpotency class* ≤ 3.

Theorem (Lutowski, Szczepański 2023)

3. Γ – *virtually abelian of minimal Hirsch length:*
$$h(\Gamma) = 15.$$

Bieberbach groups

Torsion-free virtually abelian (of rank $n \in \mathbb{N}$) group Γ

Γ — Biebierbach group

$$0 \longrightarrow L \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

$$L = \left\{ z \in \mathbb{Q}^n : \begin{bmatrix} I & z \\ 0 & 1 \end{bmatrix} \in \Gamma \right\}, L \cong \mathbb{Z}^n$$

$s \colon G \to \mathbb{Q}^n$

$G \subset \mathrm{GL}(n, \mathbb{Z})$ – finite

$$\Gamma = \left\{ \begin{bmatrix} g & s(g) + z \\ 0 & 1 \end{bmatrix} : g \in G, \begin{bmatrix} I & z \\ 0 & 1 \end{bmatrix} \in L \right\}$$

Data of a Bieberbach group

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

1. $G \subset \mathrm{GL}(n, \mathbb{Z})$ – finite.
2. $\mathbb{Z}^n$ – left G-module (matrix multiplication).
3. $\alpha = [\overline{s}] \in H^1(G, \mathbb{Q}^n/\mathbb{Z}^n)$ – a special cohomology class, where
$$\overline{s}(g) = s(q) + \mathbb{Z}^n.$$

Remark

Solvability of Γ is built into G:

$$\Gamma \text{ is solvable } \Leftrightarrow G \text{ is solvable}$$

Minimal non-solvable Bieberbach groups

Definition

Let Γ be a Bieberbach group as above. We will call Γ *minimal non-solvable* (*MNS*), if it is non-solvable and every subgroup Γ' of Γ such that

- Γ' is of smaller dimension than Γ or
- $\Gamma' = \pi^{-1}(H)$ for some proper subgroup H of G

is solvable.

Theorem (Hiller-Marciniak-Sah-Szczepański 1987, Plesken 1989)

1. *There exists a non-solvable Bieberbach group of dimension 15.*
2. *For holonomies $A_5, L_3(2)$ and $\mathrm{SL}_2(5)$ the dimension is at least 15.*

Proposition

If Γ is a MNS Bieberbach group, then $h(\Gamma) \geq 15$.

Holonomy of MNS Bieberbach groups

$G \subset \mathrm{GL}(n, \mathbb{Z})$ and we have a decomposition of $\mathbb{Q}G$-module $\mathbb{Q}^n$:

$$\mathbb{Q}^n = L_1 \oplus \ldots \oplus L_k.$$

General case: G is a holonomy of a Bieberbach group

(G1) (Hiss-Szczepański 1991) $k > 1$ and for a prime $p \mid |G|$ some component of $\mathbb{C}G$-mod. $\mathbb{C}^n$ lies in the principal p-block of G.

(G2) (Lutowski 2021) $L_1 \ncong L_k$ ($\mathbb{Q}^n$ is not homogeneous).

Specific case: G is a holonomy of a MNS Bieberbach group

(S1) G is MNS (in the set-theoretic sense).

(S2) G is perfect.

(S3) Action of G on L_i factors through MNS group, for $1 \leq i \leq k$.

(S4) If action of G on L_i factors through simple group, then $\bigoplus_{j \neq i} L_i$ is faithful.

MNS groups in $\mathrm{GL}(n, \mathbb{Z})$

Our goal:

Show that there is no MNS Bieberbach group for $n \leq 14$.

Sufficient condition:

No finite MNS subgroup of $\mathrm{GL}(n, \mathbb{Z})$ is a holonomy of a Bieberbach group, for $n \leq 14$.

How to find possible holonomy groups?

All: by (G2) and (S3) check for them in the GAP library of finite irreducible subgroups of $\mathrm{GL}(n, \mathbb{Z})$ for $4 \leq n \leq 10$.

$> 10^6$: by (S2) check in the GAP library of finite prefect groups of order $\leq 10^6$ and then do check as above with lower bound 10^6 for the order of the group.

Few details

1. Hardware: Intel Core i7-10700 + 32GB RAM
2. Software: Ubuntu 22.04 + GAP 4.12.2 (compiled from sources)
3. The subgroups were calculated up to conjugacy in the whole group:

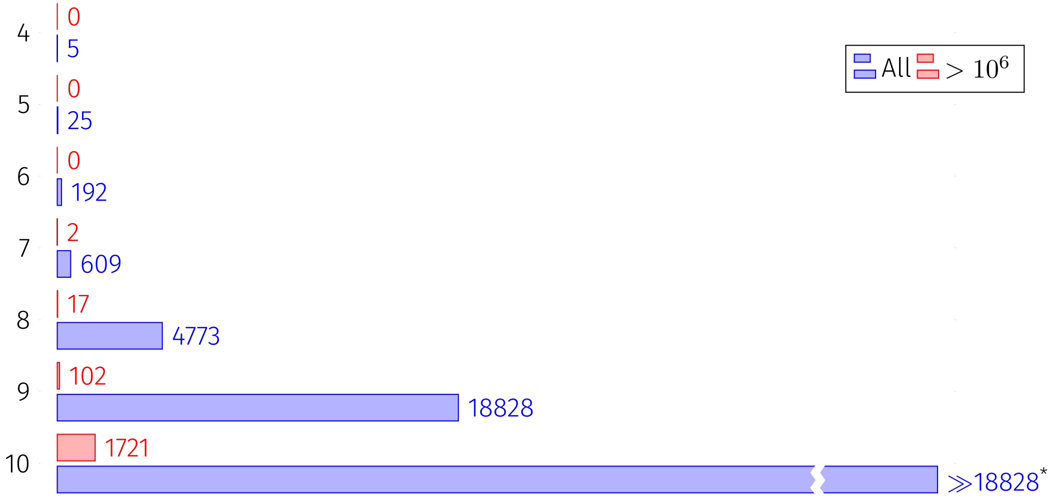
```
# Approach "All" and [Approach ">10^6"]:

MaximalNonsolvableSubgroups := function(grp[, min])
    return Filtered(
        MaximalSubgroupClassReps(grp),
        x -> [Size(x)>=min and] not IsSolvableGroup(x) );
end;
```

4. Thanks to the library, we didn't have to work with matrix groups, e.g.

```
gap> G := Image( IsomorphismPermGroup( ImfMatrixGroup(10,1,1) ) );
```

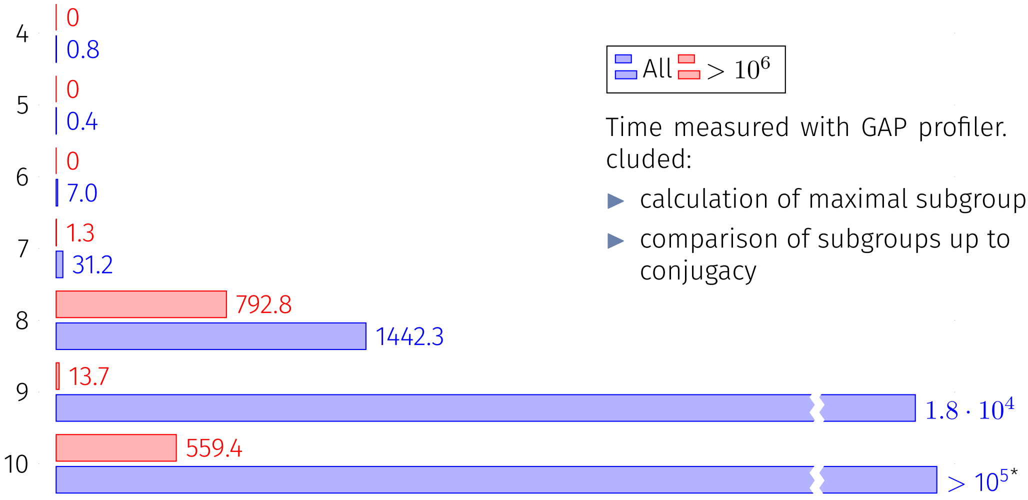
All images lie in S_n for $n \leq 270$ (median for n is 42, average ≈ 63).

Count of maximal subgroups calculation



*ran out of memory

Runtime (in seconds)



Time measured with GAP profiler. Included:

- calculation of maximal subgroups
- comparison of subgroups up to conjugacy

*ran out of memory

Some remarks and results of calculations

- Calculation of perfect groups of order $\leq 10^6$ which do not have proper non-solvable subgroups took about 1077 seconds.
- On a computer with Intel Core i7-4820K+64GB RAM the "slower" approach resulted with out of memory after about 290h of calculations.

Conclusion of calculations

1. There are 159 MNS groups of order less than or equal to 10^6 (small MNS groups).
2. Let $4 \leq n \leq 10$. There is no MNS subgroup of $\mathrm{GL}(n, \mathbb{Z})$ of order greater than 10^6.

Rational representations of MNS groups

The end of the proof

Proposition (recall)

If Γ is a MNS Bieberbach group, then $h(\Gamma) \geq 15$.

With GAP package Wedderga

The following small MNS groups have rational representations of dimension ≤ 14:

$$A_5, \mathrm{SL}_2(5), L_3(2), SL_2(7), L_2(8), L_3(2)N2^3.$$

In dimensions less than 15:

The following small MNS groups have rational reps which satisfy conditions (G1), (G2), (S4):

$$A_5, \mathrm{SL}_2(5), L_3(2).$$

Back to Hiller-Marciniak-Sah-Szczepański 1987, Plesken 1989

A Bieberbach group with holonomy $A_5, \mathrm{SL}_2(5)$ or $L_3(2)$ has dimension equal at least 15.

Thank you!