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Spin^c structures on real Bott manifolds

Joint work with Anna Gąsior

Real Bott manifolds

Real Bott tower of height n:

$$M_n \xrightarrow{\mathbb{R}\mathrm{P}^1} M_{n-1} \xrightarrow{\mathbb{R}\mathrm{P}^1} \cdots \xrightarrow{\mathbb{R}\mathrm{P}^1} M_1 \xrightarrow{\mathbb{R}\mathrm{P}^1} M_0 = \{\cdot\}$$

- $M_j \xrightarrow{\mathbb{R}\mathrm{P}^1} M_{j-1}$: projective bundle of a Whitney sum of two real line bundles.

Definition

We call M_n a real Bott manifold (RBM) of dimension n.

Remark

1. Their study involves algebra, topology, geometry, and combinatorics.
2. Can be encoded by strictly upper binary matrices.
   - Class of objects especially well suited for computer calculations.

Some examples

Example

Klein bottle $\mathbb{R}^2/\Gamma_1$

$$\Gamma_1 = \left\langle \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle$$

Example

Klein bottle $\mathbb{R}^2/\Gamma_2$

$$\Gamma_2 = \left\langle \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle$$

RBMs as orbit spaces

$$A = [a_{ij}] \in \mathbb{F}_2^{n \times n}$$

- $a_{ij} = 0$ for $i \leq j$

$M = M(A) = \mathbb{R}^n / \Gamma$

Γ is generated by

$$\begin{bmatrix}
1 & & & & & & & 0 \\
& \ddots & & & & & & \vdots \\
& & 1 & & & & & 0 \\
& & & \mathbf{1} & & & & \boldsymbol{\frac{1}{2}} \\
& & & & (-1)^{a_{i,i+1}} & & & 0 \\
& & & & & \ddots & & \vdots \\
& & & & & & (-1)^{a_{i,n}} & 0 \\
& & & & & & & 1
\end{bmatrix}$$

for $i = 1, \ldots, n$.

Spin and spin^c structures

- Both depend on the tangent bundle of the underlying manifold.
- They may, but do not have to exist.
- Their existence allows further investigation of manifolds (Dirac operators, Seiberg-Witten theory).

One-way implication

Spin structure gives rise to spin^c structure.

- Spin^c structures can be investigated even in the absence of spin structures.

Gąsior 2017, Dsouza 2018

Condition for existence of spin structures on real Bott manifold M.

Stiefel-Whitney classes

$$w_i(M) \in H^i(M, \mathbb{F}_2)$$

Orientability of M

The first Stiefel-Whitney class $w_1(M)$ vanishes.

Existence of spin structure on M

M is orientable and the second Stiefel-Whitney class $w_2(M)$ vanishes.

Existence of spin^c structure on M

M is orientable and the second Stiefel-Whitney class $w_2(M)$ is reduction modulo 2 of some element from $H^2(M, \mathbb{Z})$.

Spin^c structures on real Bott manifolds

1. Universal coefficient theorem
2. Bockstein maps associated to short exact sequences

$$0 \to \mathbb{F}_2 \to \mathbb{Z}/4 \to \mathbb{F}_2 \to 0 \quad \text{and} \quad 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{F}_2 \to 0$$

Spin^c structures on real Bott manifolds

1. Universal coefficient theorem
2. Bockstein maps
3. Commutative diagram

$$\begin{tikzcd}
 & & & H^2(M,\mathbb{Z}) \arrow[d, "\cdot 2"] \\
H^1(M,\mathbb{Z}) \arrow[r, "\cdot 2"] & H^1(M,\mathbb{Z}) \arrow[r, "\mathrm{mod}\,2"] & H^1(M,\mathbb{F}_2) \arrow[r, "\text{Bockstein}"] \arrow[dr, "\text{Bockstein}"] & H^2(M,\mathbb{Z}) \arrow[d, "\rho"', "\mathbf{mod\ 2}"] \\
 & & & H^2(M,\mathbb{F}_2) \arrow[dl, "\text{Bockstein}"'] \arrow[d, "\text{Bockstein}"] \\
 & & H^3(M,\mathbb{F}_2) & H^3(M,\mathbb{Z}) \arrow[l, "\mathrm{mod}\,2"']
\end{tikzcd}$$

» part of long exact sequence for H^1, H^2

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\end{tikzcd}$$

- part of long exact sequence for H^2, H^3

Spin^c structures on real Bott manifolds

1. Universal coefficient theorem
2. Bockstein maps
3. Commutative diagram

» diagrams combined

$$\begin{tikzcd}
 & & & H^2(M,\mathbb{Z}) \arrow[d, "\cdot 2"] \\
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Image of the map

M is spin^c $\iff w_2 := w_2(M) \in \operatorname{im} \rho$

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$$\text{M is spin}^c \iff w_2 := w_2(M) \in \operatorname{im}\rho$$

- $A = [a_{ij}]$ – the defining matrix of real Bott manifold M
- $\alpha_j := \sum_i a_{ij}x_i$ ("column sum")

Lemma (Kamishima, Masuda 2009)

$$H^*(M, \mathbb{F}_2) \cong \langle x_1, \ldots, x_n | x_1^2 = \alpha_1 x_1, \ldots, x_n^2 = \alpha_n x_n \rangle$$

- $A^{(j)}$ – jth column of A
- $S_1 := \operatorname{span}\{x_j^2 : A^{(j)} \neq 0\}$
- $S_2 := \operatorname{span}\{x_k x_l : k < l \text{ and } A^{(k)} = A^{(l)}\}$

Image of the map

M is spin^c $\iff w_2 := w_2(M) \in \operatorname{im} \rho$

- $A = [a_{ij}]$ – the defining matrix of real Bott manifold M
- $A^{(j)}$ – jth column of A
- $S_1 := \operatorname{span}\{x_j^2 : A^{(j)} \neq 0\}$
- $S_2 := \operatorname{span}\{x_k x_l : k < l$ and $A^{(k)} = A^{(l)}\}$

$$\operatorname{im} \rho = S_1 \oplus S_2$$

Spin^c structures on real Bott manifolds

Combinatorial approach

- $A \in \mathbb{F}_2^{n \times n}$ – strictly upper triangular
- $A^{(j)}$ – j-th column, $A_{(i)}$ – i-th row
- $M = M(A)$ – orientable real Bott manifold

Spin^c structures on real Bott manifolds

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- $M = M(A)$ – orientable real Bott manifold
- $A' = [a'_{ij}] \in \mathbb{F}_2^{n \times n}$ – given by

$$a'_{ij} = \begin{cases} 0 & \text{if } i \geq j \text{ or } A^{(i)} = A^{(j)} \\ \langle A_{(i)}, A_{(j)} \rangle & \text{otherwise} \end{cases}$$

Spin^c structures on real Bott manifolds

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Theorem

M admits a spin^c structure if and only if

$$A'^{(j)} = 0 \text{ or } A'^{(j)} = A^{(j)}$$

for every $3 \leq j \leq n-2$.

Code in C

```c
bool is_spinc(const uint64_t *mat, const int dim)
{
    if (!is_orientable(mat, dim)) {
        return false;
    }
    for (int j=2; j<dim-2; j++) {
        for (int i=0, e=1, z=1; i<j; i++) {
            int aij = equal_cols(mat, dim, i, j) ?
                        0 : scalar_product(mat[i], mat[j]);
            if (z && aij) {
                z = 0;
            }
            if (e && aij!=((mat[i]>>j)&1) {
                e = 0;
            }
            if (e==0 && z==0) {
                return false;
            }
        }
    }
    return true;
}
```

```c
int row_sum(uint64_t r)
{
    return __builtin_popcountl(r);
}
```

```c
bool is_orientable(uint64_t *mat, int dim)
{
    for (int j = 0; j < dim; ++j) {
        if (row_sum(mat[j]) & 1) {
            return false;
        }
    }
    return true;
}
```

```c
int scalar_product(uint64_t a, uint64_t b)
{
    return __builtin_parityl(a & b);
}
```

```c
bool equal_cols(uint64_t *mat, int dim, int i, int j)
{
    uint64_t mask = (1<<i) ^ (1<<j);
    for (int r=0; r<dim; r++) {
        if (scalar_product(mat[r],mask)) {
            return false;
        }
    }
    return true;
}
```

Application

Five-dimensional case

Corollary

An orientable 5-dimensional real Bott manifold $M = M(A)$ admits a spin^c structure if and only if

$$A_{(3)} = 0 \text{ or } a_{12} = 0 \text{ or } a_{23} = 0.$$

Application

Backtrack

- $A \in \mathbb{F}_2^{n \times n}$ – strictly upper diagonal
- $M = M(A)$ – orientable real Bott manifold
- A_1 – A after deleting first row and column

Corollary

$$M(A) - \text{spin}^c \implies M(A_1) - \text{spin}^c$$

Spin and Spin^c structures in low dimensions

- 2017: Choi, Masuda, Oum
  - Defining matrices as adjacency mats of acyclic digraphs (nauty, bliss).

Number of all, orientable, spin and spin^c real Bott manifolds (RBMs)

dim	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
RBM	12	54	472	8,512	328,416			
orient.	3	8	29	222	3,607	131,373		

Spin and Spin^c structures in low dimensions

- 2017: Choi, Masuda, Oum
  - Defining matrices as adjacency mats of acyclic digraphs (nauty, bliss).
- 2025: Gąsior, Lutowski
  - Dimension 11 possible with backtrack approach.

Number of all, orientable, spin and spin^c real Bott manifolds (RBMs)

dim	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
RBM	12	54	472	8,512	328,416	26,607,465	4,438,269,992	?
orient.	3	8	29	222	3,607	131,373	10,356,882	?
spin^c	3	7	18	57	254	1,352	10,306	105,703
spin	3	4	7	14	43	132	670	5,195

`github.com/rlutowsk/spinc-bott`

Thank you!