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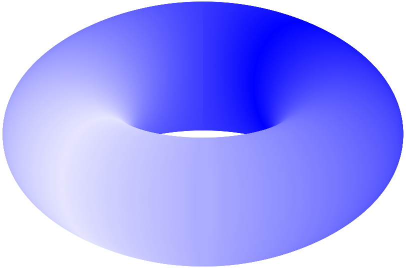
Geometry and topology of manifolds

Complex Vasquez invariant

with Anna Gąsior

Flat manifolds

Compact tori



Remark

Up to diffeomorphism one in every dimension.

Flat manifolds

Definition

1. Crystallographic group of dimension n: discrete and cocompact subgroup of $\mathrm{Iso}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$.
2. Bieberbach group: crystallographic and torsionfree.

Theorem (Bieberbach 1911)

Crystallographic group Γ of dimension n fits into the short exact sequence

$$0 \to L \to \Gamma \to G \to 1,$$

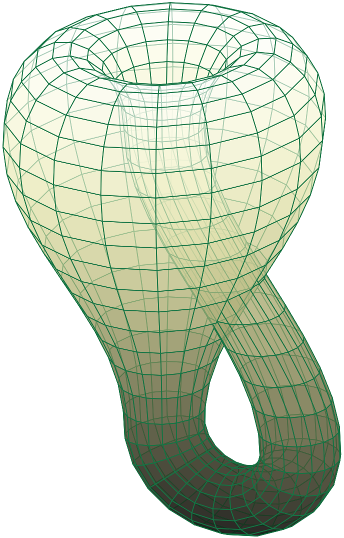
where $L \cong \mathbb{Z}^n$ – maximal normal abelian in Γ, G – finite.

Definition

Γ – Bieberbach group. We get a flat manifold

$$X = \mathbb{R}^n/\Gamma = T/G,$$

where $T = \mathbb{R}^n/L$.



Bieberbach groups and flat manifolds

Let Γ be a Bieberbach group as before:

$$0 \to L \to \Gamma \to G \to 1 \qquad (1)$$

and $X = \mathbb{R}^n/\Gamma$.

1. L is a faithful G-module, so $G \subset \mathrm{Aut}(L) \cong \mathrm{GL}_n(\mathbb{Z})$.
2. In $\mathrm{GL}_n(\mathbb{R})$, G is conjugate to the holonomy group of X.
3. $\tilde{X} = \mathbb{R}^n$.
4. X is $K(\Gamma, 1)$ space.
5. $\alpha \in H^2(G, L)$ corresponding to (1) is special:

$$\forall_{H<G} H \neq 1 \Rightarrow \mathrm{res}^G_H \alpha \neq 0.$$

Remark: enough to take $H \in \mathscr{X}$ – set of representatives of conjugacy classes of subgroups of G of prime order.

Corollary

To define a flat manifold we need a faithful G-lattice L and a special element $\alpha \in H^2(G, L)$.

Vasquez number

Vasquez number

Theorem (Vasquez 1970)

There exists a number $n(G)$ such that if L is a G-lattice with a special element α then there exists a $\mathbb{Z}$-pure submodule L' of L st.

1. $\operatorname{rk}_{\mathbb{Z}}(L/L') = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} L/L') = \dim_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} L/L') \leq n(G)$,
2. $\nu_*(\alpha)$ *is special,*

where $\nu\colon L \to L/L'$ is the natural homomorphism.

Remark

1. Vasquez constructs L' such that $\operatorname{rk}_{\mathbb{Z}}(L/L')$ is bounded by a number depending on G only.
2. Cliff and Weiss in 1990 – using completely different method – improved the bound:
$$n(G) \leq \sum_{H \in \mathcal{X}} [G:H].$$
3. We can modify the proof of Vasquez and construct L' in a way that $\operatorname{rk}_{\mathbb{Z}}(L/L') \leq \sum [G:H]$.

Examples

Theorem (Szczepański 1997)

$n(G) = 1$ iff G is cyclic of order equal to a product of different primes.

Theorem (Cliff, Weiss 1990)

If G is a p-group then $n(G) = \sum_{H \in \mathscr{X}} [G : H]$.

Theorem (Filar 2014)

$n(G) \leq 2$ if and only if $G = C_n \rtimes C_k$ with

$$G = \langle x, y \mid x^n = y^k = 1, yxy^{-1} = x^r \rangle,$$

where $k \in \{2, 4\}$, n is a product of distinct primes and $r^2 \equiv 1 \mod n$.

Consequences

Theorem

Let X' be a flat manifold with holonomy group G. Then there exists a fiber bundle

$$T \longrightarrow X' \xrightarrow{\pi} X,$$

where T is a flat torus and X is a flat manifold of dimension $\leq n(G)$.

Corollary

Characteristic classes of X' vanish in dimension $> n(G)$.

Main ingredient of the proof.

We have an exact sequence of vector bundles over X'

$$0 \longrightarrow \ker\rho \longrightarrow TX' \xrightarrow{\rho} \pi^*(TX) \longrightarrow 0$$

and $\ker\rho$ is a pullback of a bundle over X. □

Generalized hyper-elliptic manifolds

Compact complex tori

Example

Elliptic curves

$$\mathbb{C}/\langle 1, i\rangle \text{ and } \mathbb{C}/\langle 1, e^{2\pi i/3}\rangle$$

are not biholomorphic.

Remark

Up to biholomorphism we have infinitely many compact complex tori in each dimension.

Generalized hyperelliptic manifolds

Definition

Let T be a compact complex n-torus with free action of a finite group G. We call $X = T/G$ a generalized hyperelliptic or compact flat Kähler manifold.

Theorem

$\Gamma = \pi_1(X)$ is a Bieberbach group.

Theorem (Johnson, Rees 1990)

There exists a monomorphism $\iota \colon \Gamma \to U(n) \ltimes \mathbb{C}^n$ and $X = \mathbb{C}^n/\iota(\Gamma)$.

Remark

The inclusion above is given by the (almost) complex structure J of $\mathbb{R}^{2n}/\Gamma$.

Algebraic data for GHMs

Corollary

Compact flat Kähler manifold is a flat manifold with a complex structure.

$$0 \to L \to \Gamma \to G \to 1$$

1. $L \cong \mathbb{Z}^{2n}$.
2. $T = \mathbb{C}^n / \iota(L)$.
3. $J \in \mathrm{End}_{\mathbb{R}G}(\mathbb{R} \otimes_{\mathbb{Z}} L)$.

Corollary

In order to define a flat Kähler manifold we need:

1. *a faithful G-module L;*
2. *a special element* $\alpha \in H^2(G, L)$;
3. *a complex structure* $J \in \mathrm{End}_{\mathbb{R}G}(\mathbb{R} \otimes_{\mathbb{Z}} L)$.

Complex Vasquez invariant

Complex structures

Definition

Let V be a KG-module, where K is $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$.

- Complex structure on V: $J \in \mathrm{End}_{\mathbb{R}G}(\mathbb{R} \otimes_K V)$ st. $J^2 = -id$.
- A module admitting a complex structure is called essentially complex.

Remark (Johnson 1990)

Let L be an G-module. Whether L is essentially complex or not can be described in terms of properties of simple components of $K \otimes_{\mathbb{Z}} L$, where $K = \mathbb{R}$ or $K = \mathbb{Q}$.

Corollary

Let L be an essentially complex G module and let L' be a submodule of L. Then L' is essentially complex iff L/L' is.

Theorem (Gąsior, L. 2023)

There exists a number $n_{\mathbb{C}}(G)$ such that if L is an essentially complex G-lattice with a special element α then there exists a $\mathbb{Z}$-pure submodule L' of L st.

1. *L/L' is essentially complex,*
2. $\mathrm{rk}_{\mathbb{Z}}(L/L')/2 = \dim_{\mathbb{C}}(\mathbb{R} \otimes_{\mathbb{Z}} L/L') \leq n_{\mathbb{C}}(G)$,
3. *$\nu_*(\alpha)$ is special,*

where $\nu \colon L \to L/L'$ is the natural homomorphism.

Proof.

1. L'' – submodule of L from Vasquez construction.
2. There exists $L' \subset L''$ of maximal possible (and unique) rank st. L/L' is essentially complex:
$$\mathrm{rk}_{\mathbb{Z}}(L/L') \leq 2\,\mathrm{rk}_{\mathbb{Z}}(L/L'') \Rightarrow n_{\mathbb{C}}(G) \leq n(G).$$
3. $L \xrightarrow{\nu} L/L' \xrightarrow{\pi} L/L''$: $(\pi\nu)_*(\alpha)$ special $\Rightarrow \nu_*(\alpha)$ special.

□

Properties and examples

Proposition

$$n(G)/2 \leq n_{\mathbb{C}}(G) \leq n(G)$$

Proposition

Let $\mathrm{Irr}_1(G) := \{\chi \in \mathrm{Irr}(G) : \nu_2(\chi) = 1\}$*:*

$$n_{\mathbb{C}}(G) \leq \frac{1}{2}\left(n(G) + \sum_{\chi \in \mathrm{Irr}_1(G)} m_{\mathbb{Q}}(\chi)\chi(1) \right).$$

For odd order G:

$$n_{\mathbb{C}}(G) \leq (n(G)+1)/2.$$

Example

1. $n_{\mathbb{C}}(C_3^2) = n(C_3^2)/2 = 6.$
2. $n_{\mathbb{C}}(C_2^2) = 5$ (note: $n(C_2^2) = 6$).

Deformations

Holomorphic map

Definition

A homomorphism f of $\mathbb{R}G$-modules V and W, with complex structures J_V and J_W, is holomorphic if

$$fJ_V = J_W f.$$

Lemma

Keeping the above notation, assume that f is holomorphic.

1. *Kernel of f is J_V-invariant.*
2. *If f is onto, then J_W is uniquely determined by J_V.*

Corollary

Let $f\colon V \to W$ be an epimorphism of $\mathbb{R}G$-modules, with complex structures J_V and J_W. Then f is holomorphic iff $\ker f$ is J_V-invariant.

Deformations

Definition

Let L be a G-module. Denote by $\mathfrak{D}(G)$ the space of all complex structures on L:

$$\mathfrak{D}(G) := \{J \in \operatorname{End}_{\mathbb{R}G}(\mathbb{R} \otimes_{\mathbb{Z}} L) : J^2 = -id\}$$

Proposition (Gąsior, L. 2023)

Let $L' \subset L$ be essentially complex G-modules and let J be a complex structure on L. Then J can be continuously deformed in $\mathfrak{D}(G)$ to J' such that $\mathbb{R} \otimes_{\mathbb{Z}} L'$ is J'-invariant.

Remark

1. Let Γ be a Bieberbach group as before, with essentially complex holonomy representation. $\mathfrak{D}(G)$ describes all complex structures on the corresponding flat manifold.
2. Deformations preserve complex vector bundles. We do not lose informations about Chern classes.

Holomorphic tangent bundle

Theorem (Gąsior, L. 2023)

Let X' be a GH-manifold with holonomy group G. Then there exists a deformation X'_1 of X' and a fiber bundle

$$T \longrightarrow X'_1 \xrightarrow{\pi} X,$$

where T is a compact complex torus, X is a generalized hyperelliptic manifold of dimension $\leq n_{\mathbb{C}}(G)$ and π is holomorphic.

Corollary

Chern classes of X' vanish in dimension $> n_{\mathbb{C}}(G)$.

Main ingredient of the proof.

We have an exact sequence of complex vector bundles over X'_1

$$0 \longrightarrow \ker\rho \longrightarrow (TX'_1)^{1,0} \xrightarrow{\rho} \pi^*(TX^{1,0}) \longrightarrow 0$$

and $\ker\rho$ is a pullback of a bundle over X. □

Thank you!