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The R_∞ property for nilpotent quotients of free groups and surface groups

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$$R(\varphi) = \begin{cases} 2 \text{ if } n \text{ is even.} \\ 1 \text{ if } n \text{ is odd.} \end{cases}$$

Motivation from topology: A tiny bit of fixed point theory

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Let $p : \widetilde{X} \to X$ be the universal cover of X, then

$$\mathrm{Fix}(f) = \bigcup_{\tilde{f}} p\left(\mathrm{Fix}(\tilde{f})\right)$$

where $\tilde{f}$ ranges over all lifts of f to $\widetilde{X}$.

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & X \end{array}$$

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As $[A : 2A] \leq 2^{\#\mathrm{gens}}$, we have that $R(\varphi) < \infty$.

Some known cases

- There are f.g. virtually abelian groups with property R_∞.

$$\text{E.g. } G = \mathbb{Z} \rtimes \mathbb{Z}_2, \ \ \mathbb{Z} \rtimes \mathbb{Z}$$

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- There are f.g. torsion free nilpotent groups with property R_∞. Gonçalves – Wong (2009), Roman'kov (2011)
- All non-elementary Gromov hyperbolic groups have property R_∞ (e.g. free groups of finite rank > 1). Levitt – Lustig (2000), Fel'shtyn (2004)

The case of free nilpotent groups

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Theorem (Roman'kov, 2011)

If $r \neq 3$ then for $c \geq 2r$: $N_{r,c}$ has property R_∞.
If $r = 3$ then for $c \geq 12$: $N_{3,c}$ has property R_∞.

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Improvement:

Theorem (D. – Gonçalvez)

$N_{r,c}$ has property $R_\infty \Leftrightarrow c \geq 2r$.

Sketch of proof

Let N be f.g. torsion free nilpotent. Define

$$L(N) = \bigoplus_{i=1}^{\infty} \frac{\Gamma_i}{\Gamma_{i+1}} = \bigoplus_{i=1}^{\infty} \frac{\sqrt{\gamma_i(N)}}{\sqrt{\gamma_{i+1}(N)}}$$

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Proof: By taking $\mathbb{C} \otimes L(N_{r,c})$, we may assume that we are working over $\mathbb{C}$.

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(We must have $\det(A) = -1$)

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The proof finishes by taking for A the companion matrix of $p(x)$. □

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With the same techniques as for the free nilpotent group, we can show that $M_{r,c}$ has property R_∞ iff $c \geq 2r$. □

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We solved this question for fundamental groups of closed surfaces.

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$$\pi_g = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle \quad (g \geq 2)$$

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Theorem (Magnus – Karrass – Solitar, 1966)

$S \in \mathrm{GL}_{2g}(\mathbb{Z})$ corresponds to a $\bar{\varphi} \Leftrightarrow S^T \Omega S = \pm\Omega$,

where $\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$

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Eigenvalues of the matrices S

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Let $S \in \mathrm{GL}_{2g}(\mathbb{Z})$ be matrix satisfying $S^T \Omega S = -\Omega$, then the eigenvalues of S are of the form

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Important: The Magnus – Karrass – Solitar theorem also holds for Aut $\left(\frac{\pi_g}{\gamma_{c+1}(\pi_g)}\right)$.

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Let $g \geq 2$: $\pi_g/\gamma_{c+1}(\pi_g)$ *has property* $R_\infty \Leftrightarrow c \geq 4$.

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Then $c > 4$ also follows.

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(Proof ctd.)

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Then $S_g^T \Omega S_g = -\Omega$ and has eigenvalues $\lambda = 1 + \sqrt{2}$ and $-\frac{1}{\lambda} = 1 - \sqrt{2}$.

. . . □

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Important observation

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Using this we can show

Corollary

If $c \geq 2(g-1)$ then $\dfrac{\tau_g}{\gamma_{c+1}(\tau_g)}$ has property R_∞.

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Thank you!