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Tame geometry: A tribute to René Thom and Stanisław Łojasiewicz

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Effective methods for computing the topological invariants of real algebraic sets

1 Preliminaries

$F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ - a C^∞-mapping.

$$\mathcal{J} = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}$$

(the determinant of the Jacobian matrix)

$B \subset \mathbb{R}^n$ - an open bounded set such that $\partial B \cap F^{-1}(0) = \emptyset$.

For almost all $y \in \mathbb{R}^n$ sufficiently close to $0 \in \mathbb{R}^n$:

$$P = \{x \in B \mid F(x) = y\}$$

is finite and

$$\forall\ p \in P \qquad \mathcal{J}(p) \neq 0$$

Definition. *The topological degree of F with respect to B and $0 \in \mathbb{R}^n$:*

$$\deg(F, B, 0) = \sum_{p \in P} \text{sign } \mathcal{J}(p)$$

Let $\mathbb{R}$ (resp. $\mathbb{C}$) denote the field of real (resp. complex) numbers. Let V be a finite dimensional real vector space and let

$$\Phi : V \times V \to \mathbb{R}$$

be a bilinear symmetric form. Let V_+ (resp. V_-) denote a maximal subspace of V on which Φ is positive (resp. negative) definite, i.e. if $x \in V_+ - \{0\}$ (resp. $x \in V_- - \{0\}$) then $\Phi(x,x) > 0$ (resp. $\Phi(x,x) < 0$). We define

$$\text{signature } \Phi = \dim V_+ - \dim V_-.$$

We shall say that Φ is *non-degenerate* if its matrix is non-singular.

Lemma 1.1 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an $\mathbb{R}$-linear functional and let $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the bilinear form given by $\Phi(x,y) = \varphi(xy)$. Then* $\text{signature}\,\Phi = \text{sign}\,\varphi(1)$. *Moreover Φ is non-degenarate if and only if $\varphi(1) \neq 0$.*

Proof. Since φ is $\mathbb{R}$-linear then for every $x \in \mathbb{R} - \{0\}$ we have $\Phi(x,x) = \varphi(x^2) = \varphi(x^2 \cdot 1) = x^2\varphi(1)$. Because $x^2 > 0$ then

$$\text{signature}\,\Phi = \text{sign}\,\varphi(1).$$

Lemma 1.2 *Let* $\varphi : \mathbb{C} \to \mathbb{R}$ *be an* $\mathbb{R}$*-linear functional and let* $\Phi : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ *be the bilinear form given by* $\Phi(z, w) = \varphi(zw)$*. Then* $\operatorname{signature} \Phi = 0$.

Proof. Let $V_+ \subset \mathbb{C}$ denote a maximal $\mathbb{R}$-subspace on which Φ is positive definite, i.e. $\Phi(z, z) = \varphi(z^2) > 0$ for every $z \in V_+ - \{0\}$. Then $\sqrt{-1}\, V_+$ is an $\mathbb{R}$-subspace of $\mathbb{C}$ and if

$$w = \sqrt{-1}\, z \in \sqrt{-1}\, V_+ - \{0\}$$

then

$$\Phi(w, w) = \varphi(w^2) = \varphi(-z^2) = -\varphi(z^2) < 0.$$

Hence $\dim V_- \geq \dim \sqrt{-1}\, V_+ = \dim V_+$.

By similar arguments $\dim V_+ \geq \dim V_-$. Hence $\dim V_+ = \dim V_-$ and

$$\operatorname{signature} \Phi = 0. \quad \square$$

Let

$$\mathcal{B} = \mathbb{R} \oplus \cdots \oplus \mathbb{R} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C} = \bigoplus_1^m \mathbb{R} \bigoplus_1^r \mathbb{C}.$$

Then $\mathcal{B}$ is a finite dimensional $\mathbb{R}$-algebra. Let $\varphi : \mathcal{B} \to \mathbb{R}$ be an $\mathbb{R}$-linear functional. Denote

$$s_1 = \varphi(1 \oplus 0 \oplus \cdots \oplus 0),$$
$$\vdots$$
$$s_m = \varphi(0 \oplus \cdots \oplus 1 \oplus \cdots \oplus 0).$$

From previous lemmas we get

Proposition 1.3 *Let $\Phi : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ be the bilinear form given by $\Phi(f, g) = \varphi(fg)$. Then*

$$\text{signature}\,\Phi = \#\{1 \leq i \leq m : s_i > 0\} - \#\{1 \leq i \leq m : s_i < 0\}.$$

Moreover if Φ is non-degenerate then $s_1 \neq 0, \ldots, s_m \neq 0$.

Let $f_1, \ldots, f_n \in \mathbb{R}[x_1, \ldots, x_n]$, let $F_R = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ and let $F_C : \mathbb{C}^n \to \mathbb{C}^n$ be its complexification. Let

$$\mathcal{J} = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}$$

denote the determinant of the Jacobian matrix. Let $\mathcal{A} = \mathbb{R}[x_1, \ldots, x_n] \,/\, \mathrm{I}$, where I is the ideal in $\mathbb{R}[x_1, \ldots, x_n]$ generated by polynomials $f_1, \ldots, f_n$. Then $\mathcal{A}$ is an $\mathbb{R}$-algebra.

From now on we shall assume that $d = \dim \mathcal{A} < \infty$ **and that** F_C **has only non-degenerate complex roots, i.e. if** $z \in F_C^{-1}(0)$ **then** $\mathcal{J}(z) \neq 0$**.**

The next two facts generalize the Fundamental Theorem of Algebra. They follow immediately from Corollary 1 in [W.Fulton, *Algebraic Curves*], p.57.

Proposition 1.4 $\#\{z \in \mathbb{C}^n : F_C(z) = 0\} = \dim \mathcal{A} = d$

So there are d complex roots for F_C and we may assume that

$$F_C^{-1}(0) = \{p_1, \ldots, p_m, q_1, \bar{q}_1, \ldots, q_r, \bar{q}_r\},$$

where $p_1, \ldots, p_m \in \mathbb{R}^n, \quad q_1, \ldots, q_r \in \mathbb{C}^n - \mathbb{R}^n$ and $\bar{q}_i$ is the complex conjugate of q_i. Clearly $m + 2r = d$.

If $f \in \mathrm{I}$ then $f = 0$ on $F_C^{-1}(0)$. Then there is an $\mathbb{R}$-homomorphism of algebras

$$\Psi : \mathcal{A} \to \mathcal{B} = \bigoplus_1^m \mathbb{R} \bigoplus_1^r \mathbb{C}$$

given by $\Psi(f) = f(p_1) \oplus \cdots \oplus f(p_m) \oplus f(q_1) \oplus \cdots \oplus f(q_r)$. It is easy to see that $\dim \mathcal{B} = m + 2r = d = \dim \mathcal{A}$.

Theorem 1.5 *If $f = 0$ on $F_C^{-1}(0)$ then $f \in \mathrm{I}$. Hence $\Psi : \mathcal{A} \to \mathcal{B}$ is an isomorphism of $\mathbb{R}$-algebras. Thus $g = h$ in $\mathcal{A}$ if and only if $g(p_i) = h(p_i)$ for $1 \le i \le m$ and $g(q_j) = h(q_j)$ for $1 \le j \le r$.*

2 The construction of bilinear forms

Denote $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. Define $\mathcal{A}^2 = \mathbb{R}[x,y]/I_2$, where e I_2 is the ideal in $\mathbb{R}[x,y]$ generated by $f_1(x), \ldots, f_n(x)$, $f_1(y), \ldots, f_n(y)$. One may check that $\mathcal{A}^2$ is isomorphic to $\mathcal{A} \otimes \mathcal{A}$.

For $1 \leq i, j \leq n$ define

$$T_{ij}(x,y) \;=\; \frac{f_i(y_1, \ldots, y_{j-1}, x_j, \ldots, x_n) - f_i(y_1, \ldots, y_j, x_{j+1}, \ldots, x_n)}{x_j - y_j}$$

It is easy to see that each T_{ij} extends to a polynomial, thus we may assume that $T_{ij} \in \mathbb{R}[x,y]$. Define

$$T(x,y) = \det\ [T_{ij}(x,y)].$$

It is easy to see that $\mathcal{J}(x) = T(x,x)$.

Definition. The residue class of $T(x,y)$ in $\mathcal{A}^2$ is called the *Bezoutian* of $f_1, \ldots, f_n$.

Theorem 2.1 *For any polynomial $q(x)$ we have*

$$q(x)T(x,y) = q(y)T(x,y) \quad \textit{in } \mathcal{A}^2.$$

Proof. Note B_j the j-th column of $[\,T_{ij}(x,y)]$. Then

$$(x_j - y_j)B_j = \begin{bmatrix} f_1(y_1,\dots,y_{j-1},x_j,\dots,x_k) - f_1(y_1,\dots,y_j,x_{j+1},\dots,x_n) \\ \vdots \\ f_n(y_1,\dots,y_{j-1},x_j,\dots,x_k) - f_n(y_1,\dots,y_j,x_{j+1},\dots,x_n) \end{bmatrix}$$

We do not change the determinant if we add to this column a linear combination of the form

$$\sum_{k \neq j} (x_k - y_k) B_k.$$

The j-th column then becomes

$$\sum_{k=1}^{n} (x_k - y_k) B_k = \begin{bmatrix} f_1(x_1,\dots,x_n) - f_1(y_1,\dots,y_n) \\ \vdots \\ f_n(x_1,\dots,x_n) - f_n(y_1,\dots,y_n) \end{bmatrix}$$

Developing this determinant relatively to the j-th column we get an element of the ideal I_2. Hence

$$(x_j - y_j)T(x,y) = 0 \qquad \text{in} \quad \mathcal{A}^2,$$

and then $x_jT(x,y) = y_jT(x,y)$ in $\mathcal{A}^2$. Hence

$$x_kx_jT(x,y) = x_ky_jT(x,y) = y_ky_jT(x,y) \qquad \text{in} \quad \mathcal{A}^2$$

and by induction

$$x_1^{a_1} \cdots x_n^{a_n}T(x,y) = y_1^{a_1} \cdots y_n^{a_n}T(x,y) \qquad \text{in} \quad \mathcal{A}^2.$$

So the theorem is true if $q(x)$ is a monomial. One gets the general case by linearity.

Proposition 2.2 *Suppose that* $p, q \in F_C^{-1}(0)$. *If* $p = q$ *then* $T(p,q) = T(p,p) = \mathcal{J}(p)$, *if* $p \neq q$ *then* $T(p,q) = 0$.

Proof. We have already proved that $T(p,p) = \mathcal{J}(p)$. Suppose that $p \neq q$. There is a polynomial $Q(x) \in \mathbb{C}[x]$ such that $Q(p) \neq 0$ and $Q(q) = 0$. Applying the same arguments as in the proof of the previous theorem one can see that there are $h_1, \ldots, h_n$, $g_1, \ldots, g_n \in \mathbb{C}[x,y]$ such that

$$Q(x)T(x,y) = Q(y)T(x,y) + \sum_{i=1}^{n} h_i(x,y) f_i(x) + \sum_{j=1}^{n} g_j(x,y) f_j(y).$$

Since $f_1(p) = \cdots = f_n(p) = f_1(q) = \cdots = f_n(q) = 0$ then $Q(p)T(p,q) = Q(q)T(p,q) = 0$, and then $T(p,q) = 0$.

Suppose that $e_1(x), \dots, e_d(x)$ form a basis in $\mathcal{A}$. Since $\mathcal{A}^2$ is isomorphic to $\mathcal{A} \otimes \mathcal{A}$ then $e_i(x)e_j(y)$ for $1 \le i, j \le d$ form a basis in $\mathcal{A}^2$. Hence there are $t_{ij} \in \mathbb{R}$ such that

$$T(x,y) = \sum_{i,j=1}^{d} t_{ij} e_i(x) e_j(y) = \sum_{i=1}^{d} e_i(x) \hat{e}_i(y) \text{ in } \mathcal{A}^2,$$

where $\hat{e}_i = \sum_{j=1}^{d} t_{ij} e_j$.

Theorem 2.3 *$\hat{e}_1, \dots, \hat{e}_d$ form a basis in $\mathcal{A}$.*

Proof. According to Theorem 1.5, $\mathcal{A}$ is isomorphic to the product $\mathcal{B} = \bigoplus_1^m \mathbb{R} \bigoplus_1^r \mathbb{C}$. Let $E_1, \dots, E_d$ be the basis given by

$$E_1 = 1 \oplus 0 \oplus \cdots \oplus 0,\ E_2 = 0 \oplus 1 \oplus \cdots \oplus 0, \dots,$$
$$E_{m+1} = 0 \oplus \cdots \oplus 1 \oplus \cdots \oplus 0,\ E_{m+2} = 0 \oplus \cdots \oplus \sqrt{-1} \oplus \cdots \oplus 0, \dots,$$
$$E_{d-1} = 0 \oplus \cdots \oplus 0 \oplus 1,\ E_d = 0 \oplus \cdots \oplus 0 \oplus \sqrt{-1}.$$

Using Proposition 1.2 it is easy to see that elements $\hat{E}_1, \dots, \hat{E}_d$ constructed as above form a basis. Moreover, since $e_1, \dots, e_d$ are non-singular combinations of $E_1, \dots, E_d$ then $\hat{e}_1, \dots, \hat{e}_d$ are non-singular combinations of $\hat{E}_1, \dots, \hat{E}_d$, and then they form a basis.

Then there are $a_1, \dots, a_d \in \mathbb{R}$ such that $1 = a_1\hat{e}_1 + \cdots + a_d\hat{e}_d$ in $\mathcal{A}$. Hence if $p \in F_C^{-1}(0)$ then

$$a_1\hat{e}_1(p) + \cdots + a_d\hat{e}_d(p) = 1.$$

Definition Let $\varphi : \mathcal{A} \to \mathbb{R}$ be the linear functional given by

$$\varphi(f) = a_1b_1 + \cdots + a_db_d,$$

for $f = b_1e_1 + \cdots + b_de_d \in \mathcal{A}$.

The functional φ is called the *Kronecker symbol*, or the *Global Residue*, associated to $f_1, \dots, f_n$.

Lemma 2.4 *If* $p_i \in F_R^{-1}(0) \quad for \quad 1 \leq i \leq m \quad and \quad T_i(x) = T(x, p_i) \in \mathcal{A}$ *then* $\varphi(T_i) = 1$.

Proof. Since $T(x, y) = \sum_{j=1}^{d} e_j(x)\hat{e}_j(y) \quad \text{in} \quad \mathcal{A}^2$ then there are $h_k, g_k \in \mathbb{R}[x, y]$ such that

$$T(x, y) = \sum_{j=1}^{d} e_j(x)\hat{e}_j(y) + \sum_{k=1}^{n} (h_k(x, y)f_k(x) + g_k(x, y)f_k(y)).$$

Because $f_1(p_i) = \cdots = f_n(p_i) = 0$ then

$$T_i(x) = \sum_{j=1}^{d} e_j(x)\hat{e}_j(p_i) + \sum_{k=1}^{n} h_k(x, p_i)f_k(x),$$

and then $T_i = \hat{e}_1(p_i)e_1(x) + \cdots + \hat{e}_d(p_i)e_d(x) \quad \text{in } \mathcal{A}$. So

$$\varphi(T_i) = a_1\hat{e}_1(p_i) + \cdots + a_d\hat{e}_d(p_i) = 1.$$

Take $p_i \in F_R^{-1}(0)$. We have assumed that $\mathcal{J}(p) \neq 0$ for every $p \in F_C^{-1}(0)$, so $\mathcal{J}(p_i) \neq 0$. Let $t_i = T_i \,/\, \mathcal{J}(p_i) \in \mathcal{A}$. From Proposition 2.2, $t_i(p_i) = 1$ and $t_i(q) = 0$ for every $q \in F_C^{-1}(0)\ ,\ q \neq p_i$.

Let $\Psi : \mathcal{A} \to \mathcal{B}$ be the isomorphism of algebras defined before. Then $\Psi(t_i) = 0 \oplus \cdots \oplus 1 \oplus \cdots \oplus 0$, where 1 is in the i-th factor.

Let $\Phi : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ be the bilinear form given by $\Phi(f, g) = \varphi(fg)$.

Lemma 2.5 signature $\Phi = \sum_{i=1}^{m} \operatorname{sign} \mathcal{J}(p_i)$.

Proof. From Lemma 2.4,

$$\varphi(t_i) = \varphi(T_i/\mathcal{J}(p_i)) = \mathcal{J}(p_i)^{-1}\varphi(T_i) = \mathcal{J}(p_i)^{-1}$$

for $1 \leq i \leq m$. Then sign $\varphi(t_i) =$ sign $\mathcal{J}(p_i)$. Now it is enough to apply Proposition 1.3.

Let $M : \mathbb{R}^n \to \mathbb{R}$ be a polynomial, let $\varphi_M : \mathcal{A} \to \mathbb{R}$ be the linear functional given by $\varphi_M(f) = \varphi(Mf)$, let $\Phi_M : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ be the bilinear form given by $\Phi_M(f, g) = \varphi_M(fg) = \varphi(Mfg)$.

Lemma 2.6 signature $\Phi_M = \sum_{i=1}^{m} \operatorname{sign} M(p_i)\mathcal{J}(p_i)$. *If Φ_M is non-degenerate then $M(p_i) \neq 0$ for every $1 \leq i \leq m$.*

Proof. Using the same arguments as in the proof of the previous lemma one can show that $\varphi_M(t_i) = M(p_i)/\mathcal{J}(p_i)$. From Proposition 2.3,

$$\text{signature}\,\Phi_M = \sum_{i=1}^{m} \operatorname{sign} M(p_i)\mathcal{J}(p_i).$$

Moreover, if Φ_M is non-degenerate then $0 \neq \varphi_M(t_i) = M(p_i) \;/\; \mathcal{J}(p_i)$.

3 A formula for the topological degree

Let $F_R = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial mapping, let $M : \mathbb{R}^n \to \mathbb{R}$ be a polynomial and let $B = \{ x \in \mathbb{R}^n : M(x) > 0 \}$. If B is bounded and $\partial B \cap F_R^{-1}(0) = \emptyset$ then $\deg(F_R, B, 0)$ will denote the topological degree of F_R with respect to B and $0 \in \mathbb{R}^n$.

Let $\mathcal{A} = \mathbb{R}[x_1, \ldots, x_n] / \mathrm{I}$, where I is the ideal in $\mathbb{R}[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_n$. If $\dim \mathcal{A} < \infty$ then one may define bilinear forms Φ and
$\Phi_M : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ the same way as in Section 2.

Theorem 3.1 (A formula for the topological degree)) *If Φ_M is non-degenerate then $\partial B \cap F_R^{-1}(0) = \emptyset$. So if B is bounded then* $\deg(F_R, B, 0)$ *is defined and*

$$\deg(F_R, B, 0) = \frac{1}{2}(\text{signature}\,\Phi + \text{signature}\,\Phi_M).$$

We give the proof under the additional assumption that all complex roots are non-degenerate, i.e. if $p \in F_C^{-1}(0)$ then $\mathcal{J}(p) \neq 0$. We want to point out that this assumption is not necessary.

Proof. From Lemma 2.6, $M^{-1}(0) \cap F_R^{-1}(0) = \emptyset$. Since $\partial B \subset M^{-1}(0)$ then $\partial B \cap F_R^{-1}(0) = \emptyset$. According to Theorem 1.4, $F_R^{-1}(0)$ is finite. In that case

$$\deg(F_R, B, 0) = \sum_{i \in P} \operatorname{sign} \mathcal{J}(p_i),$$

where $P = \{1 \leq i \leq m \ : \ M(p_i) > 0\ \}$. From Lemmas 2.5 and 2.6 it is easy to deduce that

$$\deg(F_R, B, 0) = \frac{1}{2}(\operatorname{signature} \Phi + \operatorname{signature} \Phi_M).$$

Using the same arguments one can prove

Theorem 3.2 *Let $D \subset \mathbb{R}^n$ be an open bounded set containing all $F_R^{-1}(0)$. Then*

$$\deg(F_R, D, 0) = \operatorname{signature} \Phi.$$

Example 1. $F = (x_1^2 - x_2, x_2) : \mathbb{R}^2 \to \mathbb{R}^2$

$$I = (x_1^2 - x_2, x_2)$$

$$\mathcal{A} = \mathbb{R}[x_1, x_2]/I \simeq \mathbb{R}[x_1]/(x_1^2), \quad \dim \mathcal{A} = 2$$

$$T_{11} = \frac{(y_1^2 - x_2) - (x_1^2 - x_2)}{y_1 - x_1} = y_1 + x_1$$

$$T_{12} = \frac{(y_1^2 - y_2) - (y_1^2 - x_2)}{y_2 - x_2} = -1$$

$$T_{21} = \frac{x_2 - x_2}{y_1 - x_1} = 0$$

$$T_{22} = \frac{y_2 - x_2}{y_2 - x_2} = 1$$

$$T(x, y) = \det \begin{bmatrix} y_1 + x_1 & -1 \\ 0 & 1 \end{bmatrix} = y_1 + x_1$$

$$e_1(x) = 1, e_2(x) = x_1; \quad e_1(y) = 1, e_2(y) = y_1$$

$$T(x, y) = 1 \cdot y_1 + x_1 \cdot 1 = e_1(x)e_2(y) + e_2(x)e_1(y)$$

$$\hat{e}_1 := e_2, \quad \hat{e}_2 := e_1 = 1$$

$$T(x, y) = e_1(x)\hat{e}_1(y) + e_2(x)\hat{e}_2(y)$$

$$1 = 0 \cdot \hat{e}_1 + 1 \cdot \hat{e}_2 \;\Rightarrow\; a_1 = 0, a_2 = 1$$

$$\varphi(b_1 \cdot 1 + b_2 \cdot x_1) = a_1 \cdot b_1 + a_2 \cdot b_2 = b_2$$

$$\Phi(e_1, e_1) = \varphi(1 \cdot 1) = 0$$
$$\Phi(e_1, e_2) = \varphi(1 \cdot x_1) = 1$$
$$\Phi(e_2, e_2) = \varphi(x_1 \cdot x_1) = \varphi(x_1^2) = \varphi(0) = 0$$

$$M_\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \;\Rightarrow\; \text{signature}\,\Phi = 0$$

Theorem 3.2 $\Rightarrow \quad \deg(F_R, D, 0) = 0.$

4 A formula for the local topological degree

$F = (f_1, \ldots, f_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ – a polynomial mapping.

Suppose that 0 is isolated in $F_R^{-1}(0)$. Define

$$\deg_0(F) = \deg(F_R, \text{int } B(r), 0),$$

where $B(r) = \{x \mid \| x \| \leq r\}$ and $\{0\} = F_R^{-1}(0) \cap B(r)$.

$$Q_0 := \mathbb{R}[[x_1, \ldots, x_n]]/(f_1, \ldots, f_n),$$
$$\mathcal{J}_0 - \text{the residue class of } \mathcal{J} \text{ in } Q_0$$

Theorem 4.1 (Eisenbud-Levine, Khimshiashvili) *.*

(1) $\dim_R Q_0 < \infty \quad \Rightarrow \quad 0$ *is isolated in* $F_C^{-1}(0)$

(2) Let $\theta_0 : Q_0 \to \mathbb{R}$ *be a linear functional such that* $\theta_0(\mathcal{J}_0) > 0$. *Define*

$$\Theta_0 : Q_0 \times Q_0 \to \mathbb{R}, \qquad \Theta_0(a, b) = \theta_0(ab)$$

Then Θ_0 *is non-degenerate and*

$$\deg_0(F) = \text{signature } \Theta_0$$

Proof: (as in [5]) Additional assumptions:

(a) $f_i = g_i + x_i^{2k}, \;\; 2k > \text{degree}(g_i)$

(b) if $F_C(p) = 0$ and $p \neq 0$, then $\mathcal{J}(p) \neq 0$

$$\begin{array}{c} (a) \;\Rightarrow\; x_i^{2k} \equiv -g_i \text{ in } \mathcal{A} \;\Rightarrow\; \dim \mathcal{A} = (2k)^n < \infty \\ \Rightarrow\;\; F_C^{-1}(0) \text{ is finite} \Rightarrow\;\; 0 \text{ is isolated in } F_C^{-1}(0) \end{array}$$

$$\begin{array}{c} \varphi : \mathcal{A} \to \mathbb{R} \text{ - the Kronecker symbol} \\ \Phi : \mathcal{A} \times \mathcal{A} \to \mathbb{R}, \; : \; \Phi(a,b) = \varphi(ab) \end{array}$$

$$\mathcal{A} = Q_0 \oplus \underbrace{\mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{m-1} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$$

The $(2k)^n \times (2k)^n$-matrix of Φ (of even dimension) is of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & * & * \\ 1 & * & \cdots & * & * \end{bmatrix}$$

$$0 = \text{signature } \Phi = \text{signature } \Phi_{|Q_0 \times Q_0} + \sum_{2}^{m} \text{sign } \mathcal{J}(p_i)$$

Let $D \subset \mathbb{R}^n$ be an open bounded set containing all $F_R^{-1}(0)$. By Theorem 3.2,

$$0 = \deg(F_R, D, 0) = \deg(F_R, B(r), 0) + \sum_{2}^{m} \text{sign } \mathcal{J}(p_i)$$

$$\text{signature}\, \Phi_{|Q_0 \times Q_0} = \deg(F_R, B(r), 0) = \deg_0(F)$$

$$\varphi_0 := \varphi_{|Q_0} : Q_0 \to \mathbb{R}$$
$$\Phi_0 = \Phi_{|Q_0 \times Q_0} : Q_0 \times Q_0 \to \mathbb{R}$$

Theorem 4.2 ([5, 6, 27]) $\varphi_0(\mathcal{J}_0) = \dim Q_0 > 0$

Theorem 4.3 ([4, 5, 6, 20, 27]) *Let* $\lambda : Q_0 \to \mathbb{R}$. *Then* $\lambda(\mathcal{J}_0) \neq 0$ *if and only if* $\Lambda(a,b) = \lambda(ab) : Q_0 \times Q_0 \to \mathbb{R}$ *is non-degenerate.*

$$\varphi_0(\mathcal{J}_0) > 0 \quad , \quad \theta_0(\mathcal{J}_0) > 0$$

$$\lambda_t := t\varphi_0 + (1-t)\theta_0$$

$$\forall\ t \in [0,1] \qquad \lambda_t(\mathcal{J}_0) > 0$$

$$\forall\ t \in [0,1] \qquad \Lambda_t(a,b) := \lambda_t(ab)\ \text{ -- non-degenerate}$$

$$\text{signature } \Phi_0 = \text{signature } \Lambda_1 = \text{signature } \Lambda_0 = \text{signature } \Theta_0$$

$$\text{signature } \Theta_0 = \text{signature } \Phi_0 = \text{signature}\, \Phi_{|Q_0 \times Q_0} = \deg_0(F)$$

Example2. $F(x_1, x_2) = (x_1^2 - x_2^2, 2x_1x_2)$

$$I = (x_1^2 - x_2^2, 2x_1x_2) \subset \mathbb{R}[[x_1, x_2]]$$

$$x_2^3 = -x_2(x_1^2 - x_2^2) + (\frac{1}{2}x_1) \cdot 2x_1x_2 \in I$$

$$Q_0 = \mathbb{R}[[x_1, x_2]]/I \simeq \mathbb{R}[x_1, x_2]/(x_1^2 - x_2^2, x_1x_2, x_2^3)$$

$$x_1^2 \equiv x_2^2, \quad x_1x_2 \equiv 0, \quad x_2^3 \equiv 0$$

$$Q_0 = \{b_1 \cdot 1 + b_2 \cdot x_1 + b_3 \cdot x_2 + b_4 \cdot x_2^2\}$$

$$\mathcal{J}_0 = \det \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{bmatrix} = 4(x_1^2 + x_2^2) \equiv 8x_2^2$$

$$\theta_0(b_1 \cdot 1 + b_2 \cdot x_1 + b_3 \cdot x_2 + b_4 \cdot x_2^2) = b_4$$

$$\theta_0(\mathcal{J}_0) = 8 > 0$$

$$M_{\Theta_0} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{signature } \Theta_0 = 2$$

$$\deg_0(F) = 2$$

5 Local invariants for functions

$f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ - a polynomial

$$S(r) = \{x \mid \; \| x \| = r\}$$
$$L(r) = S(r) \cap f^{-1}(0)$$
$$A_{\pm}(r) = S(r) \cap \{\pm f \geq 0\}$$

If $r > 0$ is small enough then the topology of $L(r)$, $A_+(r)$, $A_-(r)$ is well-defined up to homeomorphism.

Theorem 5.1 (Sullivan) *The Euler characteristic* $\chi(L(r))$ *is even.*

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) : \mathbb{R}^n \to \mathbb{R}^n$$

Definition. p is a *regular* (resp. *critical*) point of f if $\nabla f(p) \neq 0$ (resp. $\nabla f(p) = 0$). Put

$$J_f = \det \left[\frac{\partial^2 f}{\partial x_i \, \partial x_j} \right] = \mathcal{J}_{\nabla f}$$

A critical point p is *non-degenerate* if $J_f(p) \neq 0$

f has an *isolated critical point at the origin* if $B(r) \cap \nabla f^{-1}(0) = \{0\}$. If that is the case then $\deg_0(\nabla f)$ is well defined.

Theorem 5.2 (Khimshiashvili) *If f has an isolated critical poit at the origin then*

$$\chi(A_-(r)) = 1 - \deg_0(\nabla f)$$

$$\chi(A_+(r)) = 1 + (-1)^{n+1} \deg_0(\nabla f)$$

$$\chi(L(r)) = \begin{cases} 0 & \textit{if } n \textit{ is odd} \\ 2(1 - \deg_0(\nabla f)) & \textit{if } n \textit{ is even} \end{cases}$$

Fix $0 < y \ll r \ll 1$. Put

$$C(f) = B(r) \cap f^{-1}(-y)$$

$$D(f) = B(r) \cap f^{-1}([-y, y])$$

(Łojasiewicz): $\chi(D(f)) = 1$

(Milnor): $\chi(A_-(r)) = \chi(C(f))$

Let g be a function having only non-degenerate critical points which is sufficiently close to f together with all derivatives of order one and two.

$$\chi(D(g)) = \chi(D(f)) \quad \Rightarrow \quad \chi(D(g)) = 1$$

$$\chi(C(g)) = \chi(C(f)) \quad \Rightarrow \quad \chi(C(g)) = \chi(A_-(r))$$

$$\Delta := \{x \in B(r) \mid \nabla g(x) = 0\}$$

.

$$1 = \chi(D(g)) = (\text{Morse theory}) = \chi(C(g)) + \sum_{p \in \Delta} \text{sign } J_g(p) =$$

$$\chi(C(g)) + \deg(\nabla g, B(r), 0) = (\ \nabla g \text{ is close to } \nabla f) =$$

$$\chi(A_-(r)) + \deg(\nabla f, B(r), 0) = (\ \nabla f \text{ has an isolated zero at } 0) =$$

$$\chi(A_-(r)) + \deg_0(\nabla f)$$

6 Local invariants for curves

$H = (h_1, \ldots, h_{n-1}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n-1}, 0)$ - a polynomial mapping
$DH(x)$ - the derivative of H at x

Suppose that $\{x \mid H(x) = 0,\ \text{rank } DH(x) < n-1\} \cap B(r) \subset \{0\}$. Then near the origin $H^{-1}(0)$ is a finite union of half-branches emanating from the origin.

b - the number of half-branches equals $\# H^{-1}(0) \cap S(r)$.

$$\Omega_H = \det \begin{bmatrix} x_1 & \cdots & x_n \\ \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial h_{n-1}}{\partial x_1} & \cdots & \frac{\partial h_{n-1}}{\partial x_n} \end{bmatrix}$$

$$F_H = (\Omega_H, h_1, \ldots, h_{n-1}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$

Theorem 6.1 (Aoki, Fukuda, Nishimura,Sun) 0 *is isolated in* $F_H^{-1}(0)$*, and*

$$b = 2 \cdot \deg_0(F_H)$$

Proof. Take $G = (g_1, \dots, g_{n-1})$ close to H such that $G^{-1}(0) \cap B(r)$ is a finite union of smooth curves transversal to $S(r)$.

$$b = 2 \cdot \#\,\text{arcs in } G^{-1}(0) \cap B(r)$$

$$F_G = (\Omega_G, g_1, \dots, g_{n-1}) : \mathbb{R}^n \to \mathbb{R}^n$$

$$x \in F_G^{-1}(0) \quad \Leftrightarrow \quad x \text{ is a critical point of } \| x \|_{|G^{-1}(0)}$$

One may assume that if $F_G(p) = 0$ then $\mathcal{J}_{F_G}(p) \neq 0$.
$\mathcal{J}_{F_G}(p) > 0 \;\Rightarrow \parallel x \parallel_{|G^{-1}(0)}$ has a minimum at p
$\mathcal{J}_{F_G}(p) < 0 \;\Rightarrow \parallel x \parallel_{|G^{-1}(0)}$ has a maximum at p

$$\deg_0(F_H) = \deg(F_H, B(r), 0) = (H \text{ is close to } G) =$$

$$\deg(F_G, B(r), 0) = \sum_{p \in F_G^{-1}(0)} \text{sign } \mathcal{J}_{F_G}(p) =$$

$$\#\text{ min of } \parallel x \parallel_{|G^{-1}(0)} - \#\text{ max of } \parallel x \parallel_{|G^{-1}(0)} =$$

$$\#\,\text{arcs in } G^{-1}(0) \cap B(r)$$

$$b = 2 \cdot \deg_0(F_H)$$

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