Higher Singular Homology Groups of Planar Sets do not Behave Anomalously

— revised version —

Andreas Zastrow

Abstract: A proof is given that for arbitrary $Z \subset \mathbb{R}^2$ all singular integral homology groups $H_k(Z)$ vanish for $k \geq 2$. The proof uses the geometric interpretation of singular chains as maps f_k from some polyhedra into Z. It then tames f_k by homotopying it appropriately, so that the tamed version can be proven to have one-dimensional image. The result then follows from a paper of Curtis and Fort. It is also indicated how, based on having f_k appropriately tamed, a bounding $(k+1)$-chain for each k-dimensional chain of Z can be constructed by showing how to extend an arbitrary map f_k that describes a cycle to some map f_{k+1}. The necessary extension facilities for either taming f or for firstly defining the codomain of f_{k+1} and then the map itself are provided for the higher k by the asphericity results of all planar sets, and by the combinatorial control of paths and of homotopies by word sequences for the small values of k, which are both results developed in the author's Habilitation thesis. Apart from that, for pointing out the potential obstructions that might generate an anomalous behaviour of singular homology groups, the paper contains a geometric interpretation of the old result of Milnor and Barratt that H_3 is non-trivial for a certain wild two-dimensional complex.

1. Introduction

We recall the following old result of Milnor and Barratt ([M–B], cf. Sect.5, esp. 5.5):
1.1 Theorem : *There exists a two-dimensional metric space which has non-trivial third (and higher) singular homology groups* (cf. 1.6(Fig.a_2)).

In contrast to this we will prove:
1.2 Theorem : *Let $X \subset \mathbb{R}^2$ be arbitrary. With H_* denoting singular homology, then $H_k(X;\mathbb{Z}) = 0$ for all $k \geq 2$* (cf. 1.4/4.8).

This follows by [C–F3].(Thm.5) directly from the following technical
1.3 Lemma: *Let $X \subset \mathbb{R}^2$ and c an at least 2-dimensional cycle of X in $H_k(X;\mathbb{Z})$.*

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Then there exists another cycle c' which is algebraically homotopic to c such that c' factors through a one-dimensional continuum (cf. 4.1/4.7).

1.4 On the structure of this paper:
The proof of Lemma 1.3 will with its preparatory definitions cover almost the entire Sections 2–4, and by this more than 50% of this paper. Theorem 1.2 will then follow rather quickly in 4.8. This method of proof reduces the problem to one-dimensional sets and the result of [C–F3].(Thm.5) or [E–K3].(Thm.2.1). To contrast this result and to justify the lengthy non-trivial proof, we geometrically interpret the above mentioned result of Milnor and Barratt in our final section (esp. 5.4–5 & 5.7). I.e. we reinterpret this result in the language and terminology that will have been previously introduced in our paper. In that course we give a complete argument that a singular $H_k(X)$ might not be trivial although $\dim(X) > k$. In our penultimate remark (5.6) we point out that this proof of the [M–B]-result and the proof of 1.2 have been motivated be the same constructions and also indicate how the proof of 1.2 could have been completed in an elementary, but self-contained way, i.e. without quotes of [C–F3] or [E–K3].

1.5 Remarks on the method of proof:
Many important standard tools of algebraic topology (e.g. van Kampen's theorem, Mayer-Vietoris Theorem, Eilenberg-Steenrod coincidence theory) require some local properties of spaces (cf. 1.6), which many planar sets do not have. Therefore we have to argue without big parts of the standard machinery of algebraic topology, and therefore we will work directly with the definitions. Also [Zas5], which proves the asphericity of all planar sets, although it will be frequently quoted in this paper, does not directly imply this result: Technically it proves that all images of maps $f : S^k \to X \subset \mathbb{R}^2$ are nullhomotopic, but images of cycles need not be nullhomotopic (cf. the example in 2.2(Fig.b)). Also via the Hurewicz-Theorem [Zas5] does not imply Thm. 1.1, since many planar sets do have non-trivial H_1. By the way, the result of [Zas5] is meanwhile also available in a shorter paper [CCZ]; however the references to the methods of proof will be to the original paper [Zas5].

1.6 Remarks on related results:
This paper paper should be viewed in connection with other recent work of Eda-Kawamura ([E–K1-4]), Cannon, Conner and Lamoreaux ([C–C1-3], [C–L]), Bogley-Sieradski ([Sier], [B–S1-2]) and of the author ([Zas1-6]), all of which is centred around the question of understanding algebraic invariants of "wild" topological spaces (i.e. spaces that are neither homeomorphic to simplicial complexes nor fulfill any of the other technical assumptions that in algebraic topology are standard assumptions for ruling out complicated local homology- and homotopy groups). The simplest of these wild spaces is the Hawaiian Earrings (see Fig.a_1); and although this space looks very much like a graph, its fundamental group is known not to be free (see, e.g. [dSm]). However, the problem with this fundamental group is that although some infinite products of rings give continuous paths, the classical group-theoretical product is always finite. This is one of the places where one gets the feeling that the classical algebraical invariants, in

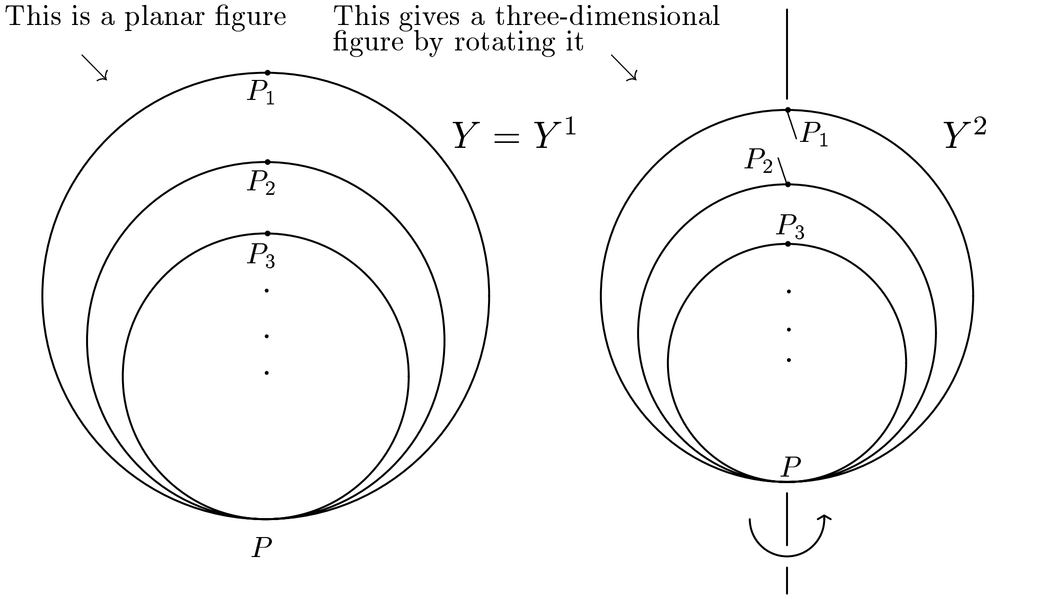

Fig. a_1 Fig. a_2

Fig.a (belonging to 1.6): This figure pictures the Hawaiian Earrings on the left hand side (Fig.a_1), and the Harmonic Balls (i.e. the two-dimensional Hawaiian Earrings which are the example of the [M–B]-result (cf. 1.1/5.1) on the right hand side (Fig.a_2). Of course, only the first three of the infinitely many rings and spheres are drawn. P, P_i denote for both pictures points of these spaces as placed in this figure.

spite of being well-defined on all topological spaces, cannot really mirror some phenomena of wild topology. Therefore, a part of the work of the above mentioned authors deals with suggesting alternative definitions of algebraic invariants of topological spaces ("Omega-groups" by [Sier], [B–S2] in order to allow some infinite products, "BIG fundamental groups" by [C–C2]). However, here in this paper only the classical definitions of algebraic invariants, in particular of singular homology groups are used.
The interest in algebraic invariants of wild topological spaces seems to be a phenomenon of the nineties. Although in the fifties and sixties there had been great interest in understanding the set-theoretic aspects of wild topological spaces, only isolated papers appeared that dealt with the behavior of their invariants: The first description of the structure of the fundamental group of the Hawaiian Earrings by Griffiths in [Gr1] (proof completed in [Gr2] and [MM]), the proof that it was not free by Higman [Higm].(Proof of Thm.6), a series of three papers by Curtis & Fort that rule out anomalous singular homology and homotopy groups for one-dimensional metric continua ([C–F1-3]), and Milnor's and Barratt's result that we already stated in 1.1. The title of this paper refers to the Milnor-Barratt paper [M–B]. At this occasion, we do also remark that the homotopy groups of the higher dimensional analogues of the Hawaiian Earring have just recently been determined up to the dimension of that space ([E–K4](Corl.1.2)).

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2. The polyhedric picture of cycles and chains

The purpose of this section is to prepare the proof of Lemma 1.3 by appropriate definitions. More precisely, the standard picture of a k-dimensional cycle in singular homology,

$$c = \sum_{i=1}^{m} k_i \cdot \sigma_i, \tag{1}$$

where $k_i, m \in \mathbb{Z}$, $k, m \geq 0$ and all σ_i are continuous maps $\Delta^k \to X$ with Δ^k being the standard k-dimensional simplex, will be replaced by a *"polyhedric picture of a chain"*, which is by definition a continuous map $f : V \to X$, where V is a *k-dimensional "almost-manifold polyhedron"*.

Such a polyhedron we define by the following axioms:

2.1 Definition:

(i) The triangulation of V is finite and consists of at most k-dimensional simplices.

(ii) All simplices of dimension k' with $k' < k$ that occur in the triangulation of V are boundary-simplices of some k-dimensional simplex.

(iii) Thinking of the triangulation of V as resulting from having to identify boundary simplices of a finite number of closed k-dimensional simplices, every $(k-1)$-face of every k-simplex is either identified with precisely one other face of a k-dimensional simplex, or it remains without identification. If the latter case does not occur in the entire triangulation, our *almost-manifold polyhedron will be called "closed"*.

(iv) Within the triangulation of V two lower-dimensional simplices will only have to be identified with each other, if this follows from the identifications of $(k-1)$-dimensional boundary faces.

(v) In the closed case (cf. (iii)), all points of V that lie outside the $(k-3)$-skeleton have the neighbourhood-type of points in a closed k-dimensional manifold. In the non-closed case, the neighbourhood-type of boundary-points of k-manifolds does also occur outside the $(k-3)$-skeleton.

(vi) V is oriented in the sense that all paths that avoid the $(k-3)$-skeleton are orientation-preserving.

(vii) V is *gallery connected*, i.e. every two points in V, if they do not belong to the $(k-2)$-skeleton, can be joined by a path in V that avoids the $(k-2)$-skeleton.

(viii) V is *locally gallery connected,* i.e. each neighbourhood of any point contains a gallery-connected smaller neighbourhood.

2.2 On the proof *of equivalence of both pictures of cycles:*
By restricting a map $f : V \to X$ to the k-dimensional simplices of V and writing the result as a formal sum, one translates a polyhedric picture into the standard one. For the inverse process one has to take the simplexes of a chain as in 2.0(1) and to identify them at those boundary sides that cancel when computing ∂c. That way the maps $\sigma_i \to X$ extend after identification to a continuous map on the polyhedron V that results from this identification process of copies of standard simplices. Without loss of generality, all $|k_i| = 1$, since summands with $k_i = 0$ can be omitted, and summands with $|k_i| > 1$ can be replaced be writing multiple times the same summand with $k_i = \pm 1$. To obtain

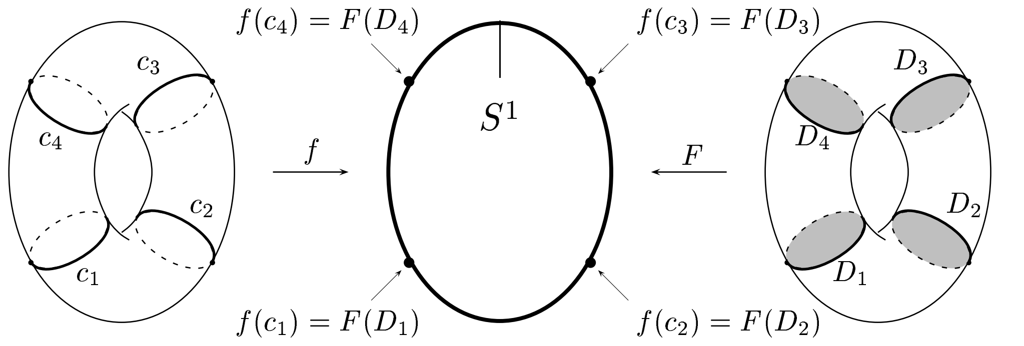


Fig.b (belonging to 2.2): This figure pictures a cycle and a bounding chain and can be used as the motivating example of this paper: The cycle is pictured on the left hand side, and its polyhedric picture is the map that squeezes a torus down to a circle. The meridianal curves of the torus are the "level-surfaces" of this map. A cone over a level-surface is a solid disk. Hence the fibre-wise cone construction (cf. 5.6), when applied to this example, gives the bounding chain of our cycle, that is on the right-hand side of this picture: a map, according to the same philosophy as on the left hand side, by applied to the solid torus.

coherent orientation (2.1(vi)) on V, the natural orientation on those standard simplices with $k_i = -1$ has to be inverted. 2.1(i)–(iv) are imposed by construction, which in case of 2.1(iii) in particular means that if same $(k-1)$-simplex should occur more than twice when computing ∂c, we choose for each summand to be cancelled one partner and only identify these "cancelling pairs". Finally 2.1(v) and 2.1(viii) drop out from the natural join structure of neighbourhoods in a simplicial complex, i.e. from the fact that a neighbourhood of interior points of each simplex σ is of type $\sigma * \mathrm{Lk}(\sigma)$. This allows on the one hand to push a path away from sufficiently lower-dimensional simplices, and it allows to determine the neighbourhood types of inner points of $(k-2)$-simplices σ: The link $\mathrm{Lk}(\sigma)$ is a graph in this dimension, by 2.1(iv) it is connected, and by 2.1(iii) the valencies of its vertices can only be one or two. Therefore it can only be an interval I or a one-sphere S^1, so that $\sigma * \mathrm{Lk}(\sigma)$ is in any case a ball, with the interior points of σ lying in the interior of this ball in the second case and on the boundary in the first case. This guarantees the correct neighbourhood type also for $(k-2)$-simplices in both cases of 2.1(iii). That way we get the non-trivial part of the claim 2.1(v) and do also see that *polyhedric pictures of closed cycles can be constructed on closed almost-manifold polyhedra*. Observe that the Axioms 2.1(i)–2.1(vi) & 2.1(viii) are also built in such a way *that those simplices of V which are in the sense of 2.1(iii) not identified with any other simplex, form a $(k-1)$-dimensional almost-manifold polyhedron which we will denote by ∂V. Obviously the restriction $f|_{\partial V} : \partial V \to X$ is a polyhedric picture of the boundary cycle ∂c, if f was a polyhedric picture of c.* Accordingly the task to prove that a cycle is nullhomologous can be viewed as an extension-task (cf. Fig.b, 5.6).

2.3 Remark: Observe that we have not been able to establish 2.1(vii), i.e. connectedness by the constructions as described in 2.2. However, since all constructions that will have to be performed in this paper to almost-manifold polyhedra V can be performed to each connected component separately, we will be able to assume this property in the remainder of this paper, also.

3. Some lemmata on paths in the plane

3.1 Remark: In [CCZ].§2 a method was proposed to parametrize each path "by oscillation" i.e. in a canonical way that has some continuity properties and guarantees a finite parameter interval even for wild (e.g. non-rectifiable, space-filling, ...) curves. This parametrization only depends on the initial segment and is independent of the given parametrization. Vice versa, it has no compatibility with inverting a path or cutting off its initial segment, and there is little chance to compute it in any concrete situation.

3.2 Definition: Let $u, v : [0,1] \to X \subset \mathbb{R}^2$, $u(0) = v(0)$ be two paths without constant segments. Let $\phi : [0,a] \to [0,1]$ and $\psi : [0,b] \to [0,1]$ be monotone functions such that $u \circ \phi$ and $v \circ \psi$ are parametrized by oscillation (cf. 3.1), and define $c := \sup\{t \mid (u \circ \phi)(s) = (v \circ \psi)(s)\ \forall s \le t\}$. Then we call $u(\phi(c)) = v(\psi(c))$ the *"falling apart point"* of u and v. Note that this definition is applicable to all paths independently of how they do intersect or self-intersect.

3.3 Remark: Let $u, v, \phi, \psi, a, b, c$ be as in 3.2. *Then $u_{|[0,a]} = v_{|[0,b]}$ as abstract paths. In addition, if ever we have an equation $u_{|[0,a']} = v_{|[0,b']}$ as abstract paths, then $a' \le a$ and $b' \le b$ with $a' = a \iff b' = b$.* This all follows straightforwardly from the fact that the parametrization by oscillation does only depend on the behaviour of the initial segment of a path as an abstract path.

3.4 Remark: Recall that by 1.3/2.2–3 we are given a polyhedric picture of a cycle $f : V \to X$ and are supposed to find a homotopic better map $f' : V \to X$. By a worst-case scenario we can assume, without loss of generality, that $X = f(V)$, since this gives us the least chances for defining f'. Vice versa, such subsets X are Peano continua; we can therefore can, without loss of generality, assume this for X. In [Zas5].(Thm.B) it was shown that every planar Peano continuum can be homotopy equivalently embedded into a partially filled modified Sierpiński Carpet (the terminology is from [FiZa]). *Therefore we can (and will) in the remainder of this paper assume, without loss of generality, that X is a partially filled modified Sierpiński Carpet,* i.e. a space that results from the Sierpiński Carpet by firstly arranging the holes in a different pattern, cf. [Zas5].A.3.3 and [FiZa].(Rem.14) (this is, what is called a "modified Sierpiński Carpet"), and then disks are glued into an arbitrary subset, (possibly into none or all) of the holes. The original and the modified Sierpiński Carpet both fulfill the characterization of [Wh] and are therefore homeomorphic to each other. A "partially filled modified Sierpiński Carpet" we will for the sake of brevity also call a *"Sierpińskian Carpet"* in the remainder of this paper. By assuming that X is a Sierpińskian Carpet it will also follow that there is, apart from 3.1, a technically different way to establish short connections, which in particular has the properties 3.6(x), (xi) & (xiii) that we will need in the remainder of this paper. The next two items are devoted to explain how these properties follow out of the results of [Zas5].

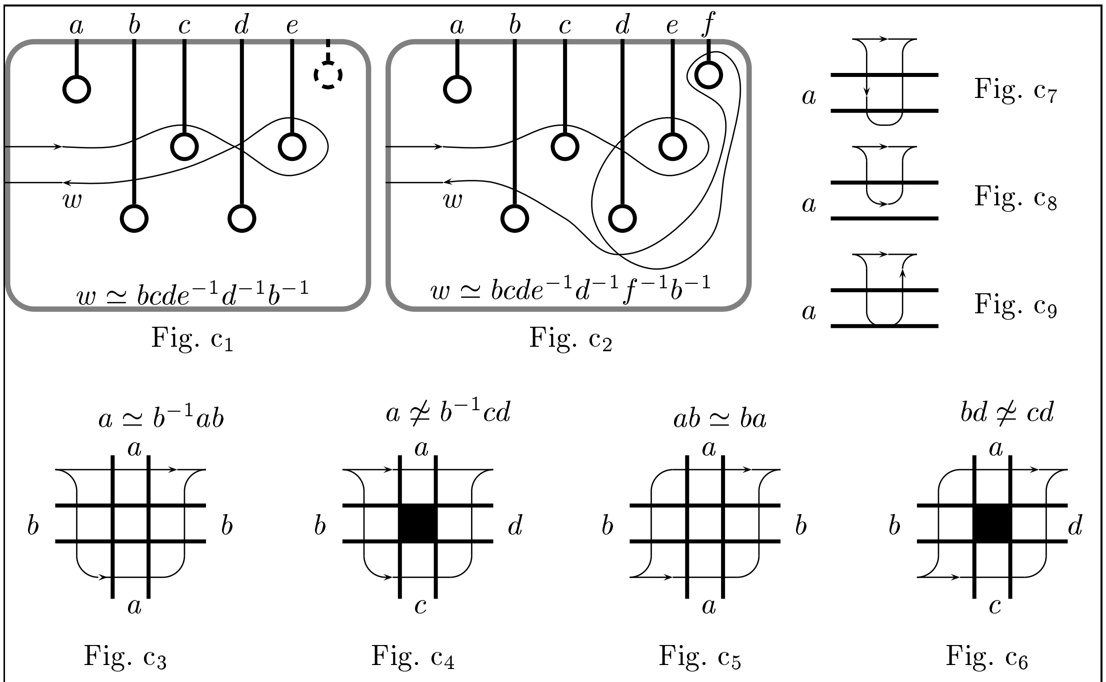


Fig.c (belonging to 3.6): This figure illustrates the standard scheme of associating strings of letters to path segments according to the way how these paths cross curves (Fig.c_1–c_2) or corridors (Fig.c_3–Fig.c_9). See the appropriate items of 3.6 for comments on which of these subfigures belong to what situation. In Figs.c_7–c_9 the horizontal curve segments are always associated with the empty string, while in the three cases for the other curve segment we get $a^{-1}a$, the empty string or "an ambiguous reading", respectively (i.e. we can have both of the two latter possibilities).

3.5 On [Zas5]: Unfortunately, [Zas5] is not a very convenient easy accessible source due to the combinatorial method that the paper chose to define *"algebraically shortest paths"*. Classical path-length could not be used, since wild spaces have many homotopy classes where each path is non-rectifiable, and also parametrization by oscillation does not seem to be appropriate, since it is not really well-behaved (cf. 3.1) making it difficult to prove concrete properties. Even existence, uniqueness and independence of the defining system of dense corridors are currently not available in literature for short paths in that sense. Therefore, in adapting a method from [Zas1&3], [Zas5] chose a combinatorial method to construct such short connections by firstly translating the problem into a problem for combinatorial objects, i.e. the *"word sequences"*, and this is what made this paper so long. On the other hand, [Zas5] does not contain a systemetic development of the theory, but it restricts itself to developing those properties that are needed for the main asphericity result, Thm.A of [Zas5]. Therefore the properties 3.6(x), (xi) & (xiii) cannot be quoted as lemmata directly from [Zas5], but it needs some basic understanding of the concept and terminology of [Zas5], in order to understand why these properties drop out of the methods developed there. This, together with some intuitional arguments for the behaviour of short paths in the plane is provided by the following:

3.6 Summary of the quotes that we need from [Zas5]:

[Reference numbers that start with letters refer to [Zas5].]

(i) It is consistent with our intuition that in a subset of the plane a "shortest connection in a given relative homotopy class for any two points should exist: If a string indicates the position of a path, then pulling at the string until it is completely hung up by holes should find this short position. However, the technical intuition for [Zas5] was that, even in a wild subset of the plane, fundamental groups should at least be almost free and therefore produce by reduction mechanisms shortest connections.

(ii) First observe that the natural way for assigning a combinatorial object to a path would be to connect each hole to infinity by a system of "reading curves" and to associate with each path a "word" that keeps a record which of these curves are crossed in which direction and order. This is the standard way of obtaining a concrete basis of the free fundamental group in case of only finitely many holes (cf. Fig.c_1–c_2).

(iii) In [Zas5] this concept had to be altered in three ways (cf. C.1.9/C.2.1):

(1.) The reading curves had to be thickened and have then been called "corridors" (A.4.6(i)). This was done to avoid that a curve can make our words infinite by crossing a reading curve as often as the graph of $f(x) = x \cdot \sin(\frac{1}{x})$ crosses the x-axis. By continuity, each corridor can only be crossed a finite number of times. Of course, having thick reading curves causes a lot of ambiguity (C.1.3). E.g. the concept of truncation (cf. B.2.2(iii)/D.1.1(iii)) needed to be invented in order to appropriately handle the case when a path ends in the middle of a corridor and therefore only half-crosses a corridor.

(2.) In difference to our Fig.c_1–c_2, we consider "lattices" (B.1.1) of horizontal and vertical corridors. The idea of this was, to be able to determine the position of a path up to a square in the lattice by the combinatorial information, and to avoid that bringing an additional hole into the picture can make a path flip around as much as Fig.c_1–c_2 show.

(3.) In order to always have finite words, it is not possible to look at all holes simultaneously from the very beginning. Instead of this we look only at the pictures that are obtained by ignoring all holes in a Sierpińskian Carpet that do not reach a certain size, and we work on the finitely punctured planes that are caused by the finitely many big holes of bigger size, only — but at all such planes simultaneously. That way we have to work with infinitely many lattices of corridors which we call a "network" (B.1.1). In each such lattice a path induces a word, and together they form a "word sequence" (A.4.6/B.1.4). *We will use ω as an operator which (up to some ambiguity) assigns such a word sequence to a given path* (cf. B.1.6(1)).

(iv) The first task of [Zas5] was understanding when an appropriate arrangement of letters actually as a word sequence describes a continuous path. After translating the natural geometric restrictions into a finite set of conditions that either affects each word separately or two adjacent words in a sequence, one can start drawing a sequence of paths, one in each finite lattice — and such paths can be proven to uniformly converge to a limit path that "realizes our word sequence" (B.1.6(2)/C.3.1).

(v) The second task then was to understand on the combinatorial level when two word sequences describe relatively homotopic or even the same path (the latter happens due to the ambiguity of (1.) and in case of tangencies to corridor boundaries, cf. Fig.c_9). These equivalence relations all are induced just by the natural relations whose geometrical origin has been pictured in Fig.c_3, c_5 & c_7–c_8. Of course, the corresponding pushing-processes of paths can only be performed, if such processes can be simultaneously applied to all words of such a sequence (B.2.1(ii)/D.1.2). But with a similar philosophy as in (iv), two finite sets of rules for how to transform word sequences were first set up and then it was proven that the so defined "weak equivalence relation" (B.2.9(i)) corresponds to the relative homotopy of paths (D.12.1), and the "strong equivalence relation" (B.2.9(ii)) relates only different word sequences describing of the same path (D.12.3).

(vi) One outcome of this discussion is actually the basis of the entire combinatorial theory: namely that a path in X will only be non-nullhomotopic, if and only if it is non-nullhomotopic in some finitely punctured plane, already. The proof is via the corresponding equivalence for nullhomotopic paths; the non-trivial part of the proof is a construction that turns the information how a word-sequence can be cancelled to the empty one into a sequence of approximating null-homotopies that uniformly converge (cf. D.8.20, [FiZa].Thm.16)

(vii) In order to prove the existence of unique reduced word sequences as a prerequisite for defining our "short paths" (cf. 3.5/3.6(i)), it was essential that the elementary equivalences had been set up so that any non-reduced word sequence could by applying "shortening" operations only be transformed into (one of the) reduced word sequences of its weak equivalence class. In particular this holds for those word sequences that describe null-homotopic paths which by shortening processes only can be transformed into the empty word sequence (B.2.12(iii)/D.9.1).

(viii) In order to reach this goal also transpositions (cf. B.2.2(ii)/Fig.c_6) had to be subdivided into enlarging and shortening ones. The convention that has been chosen in D.3.10 essentially was that, as long as either monotony is preserved or you get obstructed by holes, pushing a path towards the bottom left or towards the bottom right corner makes it shorter. I.e. our short paths (that we therefore will call "algebraically shortest paths") will not coincide with the Euclidean shortest paths, since algebraically going over the hypotenuse is not the shortest way, but going over the other sides. However, this convention had a fairly easy description on the combinatorial level (cf. B.2.3(1)).

(ix) With the same motivation as (vii) and since by (viii) some transpositions are not of shortening type, "far-distance cancellations" had to be invented (cf. Fig.c_3, B.2.2(iv)/B.2.4). I.e. we had to allow to also cancel two inverse letters in that situation where they have not been in adjacent positions, but where all letters between them belonged to such corridors that intersect with the corridor of the letters to be cancelled. Together with transpositions (which are only permitted under similar conditions), these are the processes that allow to slide a path over a region of X that is free of punctures — while classical cancellation can pull back a path only in a situation when in runs through thorns (Fig.c_7–c_8).

(x) That way it could be proven in [Zas5] that, with adding the phenomena of having to perform all processes simultaneously for all words of a word sequence and with extending the calculus to a calculus of partially commuting letters that also allows some transpositions and far-distance cancellations, the essential phenomena from the calculus of letters in a finitely generated free group still hold in this context. In particular it could be deduced (cf. D.7.9) that there is a "firm" subclass of reduced word-sequences in each weak equivalence class. *"Firm equivalence"* (B.2.9(iii)) is an equivalence relation that is stronger than "strong equivalence", and hence this implied, that, even if the describing word sequence was not entirely unique, the associated paths of reduced word sequences were unique. This defined "algebraically shortest paths".

(xi) The calculus of word sequences also provided the framework for proving another property of short paths that seems absolutely natural when viewed from the picture with which we started the discussion in (i): What happens if we pull a string into its straight position until it is hung up by holes? – In case of three connections of each pair of a trio of points whose concatenation is a nullhomotopic path, the picture should look as in Fig.d_1, or, if we take (viii) into account, as in Fig.d_2. In particular those segments where one path does not coincide with any of the connections to the third point should be monotonic and together give a triangle ("inner triangle") that is embedded into the plane and bounds a Jordan curve. The calculus of word sequences provides a framework to deduct this fact according to the scheme that we describe in the following items:

(xii) The basic framework is a discussion how the concatenation of two reduced word sequences ω and ω' might cancel to a reduced word sequence ω'' (D.11.10). One of the findings of looking at all cases is, that only two letters which are taken from our different word sequences ω and ω' might be affected by a cancellation (D.11.6) or by a transposition process (D.11.9). The discussion also subdivided the word sequences into three subword-sequences according to $\omega = \omega_1\omega_2\omega_3$ and $\omega' = \omega_1'\omega_2'\omega_3'$ by the definition that inside ω_3 and ω_1' both types of letters (i.e. those coming from crossing with horizontal and those where vertical corridors have been crossed) are affected, inside ω_2 and ω_2' just one type of letters is affected, and ω_1 and ω_3' will be those letters that we also find in the same order in ω''. The discussion further gave that ω_3 will letter-by-letter cancel against ω_1', while the ω_2- and ω_2'-parts will be only affected by transpositions and far-distance-processes. I.e. these are precisely those processes that move the concatenation of two sides of our inner triangle into a new short position, i.e. onto the third side. The fact that these sides are monotone can now be seen essentially as follows:

(a) We use the operator "$\overline{(\ )}$" to denote the result of the reduction process. If $\overline{\omega_2\omega_2'}$ results from $\omega_2\omega_2'$ only by transpositions, then ω_2 can have only contained one type of letters, and ω_2' the other. For, letters of the same type never commute, and D.11.9 does not provide any other chance how letters from two different short words can mix without cancelling first. This also means, that any non-monotony within ω_2 or ω_2' would allow direct cancellation and hence cannot occur in a short word. This monotony will be preserved if the letters of ω_2 and ω_2' mix inside $\overline{\omega_2\omega_2'}$, since the relative order of each type of letters is preserved, because letters of the same type never commute.

(b) If $\overline{\omega_2\omega_2'}$ results from $\omega_2\omega_2'$ by far distance-cancellations only, then D.11.10(2.) implies that only letters of one type will disappear from these words (note, that what we are introduced as "type" at the beginning of (xii), was called "basic direction" in [Zas5]). Consequently, since the entire word sequences ω_2 and ω_2' are supposed to be affected by these processes, $\overline{\omega_2\omega_2'}$ can only consist of the other type of letters and, as it is natural for such horizontal or vertical paths, any non-monotony would immediately allow further cancellations. But then, according to the basic subdivision above in (xii), these parts of our word sequences, would not belong to $\omega_2\&\omega_2'$, but to ω_3 and ω_1'. Hence there can also not be any non-monotony in this case.

(c) Now assume that both types of processes occur. By D.11.10.(2.)–(3.) we have to subdivide or word-sequences according to $\omega_2 = \eta_1\eta_0$ and $\omega_2' = \eta_0'\eta_1'$, such that $\eta_0\eta_0'$ is precisely the domain where the far-distance cancellations take place, while the remaining parts will only be affected by transpositions. Consequently, in $\overline{\eta_0\eta_0'}$ there will be only one type of letters, and $\eta_1\&\eta_1'$ have to consist of different types. Without loss of generality, let in η_1' be the type of letters that was cancelled, and in η_1 the one that remained. Consequently η_0' must be empty, since otherwise D.11.9 would obstruct any transposition. However, thus having only horizontal and vertical segments in $\eta_1\overline{\eta_0\eta_0'}\eta_1'$, makes any non-monotony impossible with the same arguments as given in (a) and (b).

(xiii) Observe that as result of the previous discussion we actually obtain that *each path-segment that is associated with $\overline{\omega_2\omega_2'}$ and hence any side of an inner triangle is monotonic with respect to the x- and with respect to the y-coordinate.* This is in line with Fig.d_2, and a stronger result than we got in case of the naive picture Fig.d_1.

(xiv) We have been fairly detailled in the above discussion of (xii), since such a strong monotony-result is actually not contained in [Zas5]. The monotony-result of [Zas5] is E.4.13(ii), and this is just a special case of what we have here. Namely, [Zas5] only looks in this context at triangles that have one short side inside a small square-shaped neighbourhood Q_η of edge-length 2η, and it discusses only the monotony of the two other long sides, as long as they are outside Q_η. It also makes use of the fact that the third triangle-side can only contribute letters from the bounded Q_η-domain into the cancellation process. However, the proof of E.4.13(ii) in E.4.7–E.4.12 (which is, in difference to (xii), set up as an indirect proof) can serve as a guideline how one can, based on the basic arguments that were provided in (xii), phrase a complete proof for word sequences that also takes care of some aspects that we neglected above (e.g. of having to look at infinitely many words simultaneously, cf. (iii(3.)). In 3.7(Fig.h) we give an extra graphical argument, why the weak monotony conditions from Fig.d_1 (up to two intervals of monotony for each triangle side) are not compatible with our convention of algebraically short curves (cf. (viii)).

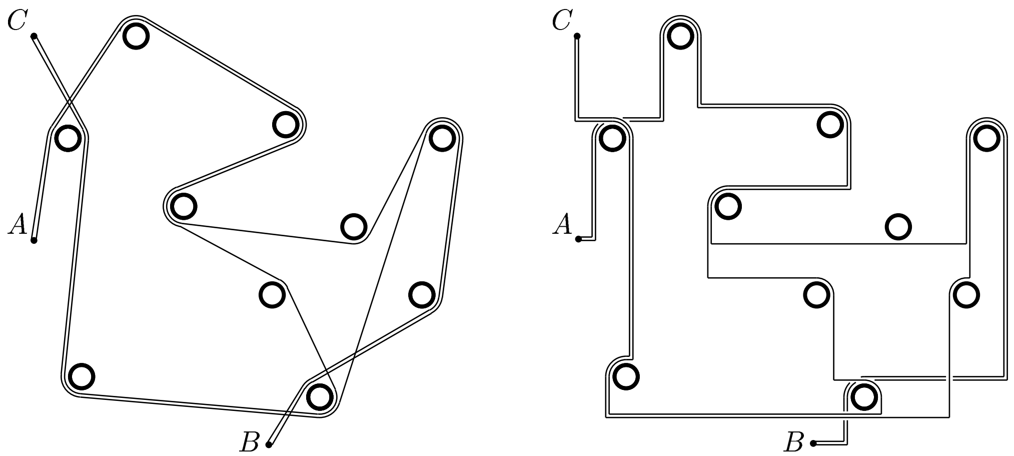


Fig. d_1 Fig. d_2

Fig.d (belonging to 3.6(xi)): Both subfigures show three paths which connect each pair of a trio (A, B, C) of points in a subset of a plane so that the concatenation of all three paths is nullhomotopic. The figure demonstrates that if these paths should be in a short position, then appropriate initial segments of the two paths that emanate from the same point should coincide, and that the non-coinciding segments should form an "inner triangle" which has only monotonic sides and is as a whole embedded into the plane without self-intersections. The left subfigure (Fig.d_1) works with short paths in the standard sense, the right subfigure (Fig.d_2) with algebraically short paths in the sense of 3.6(viii).

The other missing claim from (xi), namely that we obtain a Jordan curve for our inner triangle will be in a more technical language given as part of the proof of the following lemma:

3.7 Lemma: *Let X be a Sierpińskian Carpet and A_1, A_2, A_3 be three different points in X. Let $a_1, a_2, a_3 \in S^1$ and $f : S^1 \to X$ be a nullhomotopic path with $u(a_i) = A_i$ $(i = 1, 2, 3)$. Further assume that $f|_{[a_i,a_{i+1}]}$ is an algebraically shortest path in the sense of 3.6(vii)–(viii) $(i = 1, 2, 3 \; , \; A_4 := A_1 \; , \; a_4 := a_1)$. Then f factors according to $f : S^1 \xrightarrow{g} T_3^{\triangle} \xrightarrow{h} X$ through $T_3^{\triangle}$, or through one of those of its degenerations that are up to permutations of the indices 1,2 & 3 pictured in Fig.e. With notations as introduced in this figure, $g(a_i) = Q_i$, where g maps the segments $[a_i, a_j]$ monotonically to the shortest edge-paths in $T_3^{\triangle}$ that connect Q_i and Q_j. Further, $h(\delta)$ is a Jordan-curve whose interior belongs to X.*

Proof: In continuing the discussion from 3.6(xi)–(xii) we use square brackets to denote the segments of our one-sphere, $[a_i, a_j, a_k] := [a_i, a_j] \cup [a_j, a_k]$. Since by assumption $f([a_1, a_2, a_3])$ is homotopic to $f([a_1, a_3])$, the product $\omega(f|_{[a_1,a_2]}) \cdot \omega(f|_{[a_2,a_3]})$, must be weakly equivalent (cf. 3.6(iii(3.)), (v) & (vii)) to $\omega(f|_{[a_1,a_3]})$. According to 3.6(xii) these word sequences split up into three phases, where the end-phase of $\omega(f|_{[a_1,a_2]})$ is inverse to the start phase of $\omega(f|_{[a_2,a_3]})$. Accordingly the paths are also inverse to each other, and that way we find the segment which g shall map to t_2. Similarly those segments of

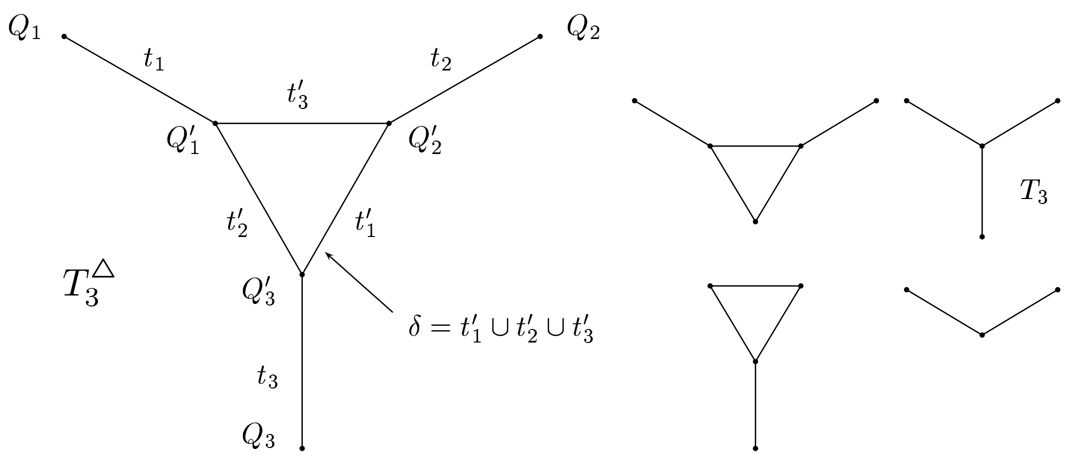
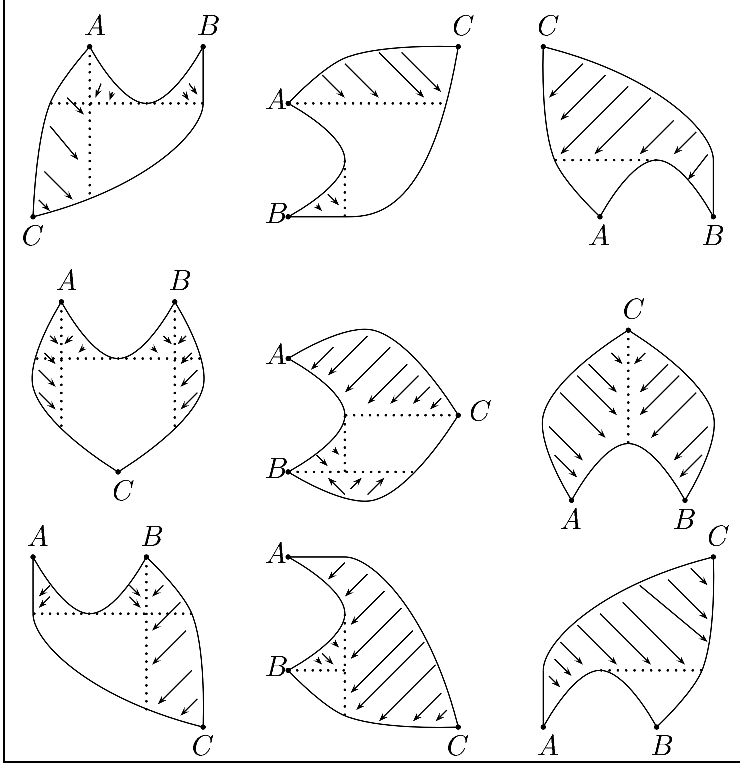


Fig.e (belonging to 3.7): The big subfigure on the left is the graph $T_3^{\triangle}$ with all its standard notations that are used in this paper. The remaining four subfigures show, up to the natural symmetries of $T_3^{\triangle}$, those degenerations that still can occur when this graph serves as a factoring-through graph in the sense of 3.7, i.e. in particular for three distinct points A_1, A_2, A_3. The labelling of $T_3^{\triangle}$ extends to these degenerated graphs as well, with the understanding that edges might be degenerated to points, and points with different labels might coincide.

S^1 are determined which are mapped to t_1 and t_3. On the remaining segments of S^1 our map g is adjusted to map appropriately onto the remaining edges t'_1, t'_2 and t'_3 of $T_3^{\triangle}$. Having completed the definition of g, the map h is uniquely determined by $f = h \circ g$. Since $\omega(h|_{t'_1})$, $\omega(h|_{t'_2})$ and $\omega(h|_{t'_3})$ are identical with subword sequences of $\omega(f|_{[a_i,a_j]})$ (or with their inverses), they are reduced already and, since each of them results according to 3.6(xii) by reducing a product of the two other ones (or their inverses) by far-distance cancellations and transpositions only, by 3.6(xii(a))–(xiv) they satisfy monotony conditions. Hence if two of our three segments should intersect each other, subsegments of these curves together would give a Jordan curve. Such subsegments are naturally bounded on the one hand by our intersection points, and on the other hand either by the next intersection point, or by the natural common endpoint of these segments. This Jordan curve bounds a cell, and the interior of such a cell can either be free of holes or have holes. In case it is free, we have a contradiction to the uniqueness of the reduced path between the two intersection points. If it should not be free of holes we have no chance to monotonically continue these segments to some end-points in the plane and to connect them by a third segment in such a way that a nullhomotopic curve results. However, by assumption, the concatenation of all three segments was nullhomotopic and has to remain nullhomotopic after the cancellation of those segments that g maps to the t_i. This proves that f maps the union of these segments to a Jordan curve whose interior has to be free of holes. $\square$

3.8 Definition of the equivalence relation "$\sim$": Let $f : V \to X$ be a polyhedric picture of a cycle such that f takes different values on neighbouring vertices of V. Let e_1 and e_2 be two edges with the same start point P, and assume that $f|_{e_1}$ coincides with $f|_{e_2}$ for some initial segment. Let e'_1 and e'_2 be these initial segments of e_1 and

Fig.f (belonging to 3.6(xiv)/3.7): The purpose of this figure is to show that inner triangles as they were obtained in 3.6(Fig.d) always have to satisfy that all three sides are monotonic with respect to the x- and to the y-coordinate — in difference to the naive picture 3.6(Fig.d$_1$), where we obtained with respect to one of the coordinates up to two intervals of monotony. Therefore the figure starts in the upper left subfigure with one triangle-side $\overline{A\ B}$ which is U-shaped and hence has this behaviour. It then tries to complete this curve to a Jordan curve that might be an inner triangle in the sense of 3.6(xi). There are essentially three possibilities, namely the third vertex C could have an x-coordinate lower than that of A, bigger than B or between those of A and B. All these possibilities are drawn in the first column of this figure.
Rotating our curve $\overline{A\ B}$ from the first column by multiples of 90° gives the three other possibilities for a how a curve with the monotony conditions from 3.6(Fig.d$_1$) could distribute its directions and intervals of monotony between the x- and the y-coordinate. Taking into account that even having the selected pushing-directions from 3.6(viii) leaves our plane with a left/right-symmetry, only two of these three possible rotations remain to be considered as essentially different cases. They are considered in the second and third column of this figure.
Each of these subfigures indicates, that our conventions for algebraically shortest paths according to 3.6(viii) and the fact that by 3.7 the interior of the triangle is free of holes, make it impossible to have the vertices of this inner triangle were they are supposed to be — since vertices of inner triangles are defined as falling apart points (cf. 3.2).



e_2, respectively. By 3.3, the endpoints of e_1' and e_2' are clearly associated with each other, but also each interior points $x \in e_1'$ is clearly associated with some interior point $y \in e_2'$ by the relation that $f|_{[P,x]} = f|_{[P,y]}$ as abstract paths. This defines a bijection $e_1' \to e_2'$. We define for all such edges that belong to one triangle $\in V^{[2]}$ the corresponding bijections. And finally we define *two points of the one-skeleton* $V^{[1]}$ *to be equivalent,* if they can be mapped onto each other by a finite chain of these bijections.

3.9 Lemma: *Assume that the restriction of $f : V \to X$ to the boundary curve of each two-simplex Δ_i factors through a $T_3^{\triangle}$-graph in the sense of 3.7, in formulae:*

$$f|_{\partial\Delta_i} : \partial\Delta_i \xrightarrow{g_i} T_3^{\triangle} \xrightarrow{h_i} X.$$

Then the domain of $\mathbb{R}^2$ that lies on the boundary or in the interior of at least one of the closed curves $h_i(\delta)$ can by straight-line segments and segments of the curves $h_i(t_j')$

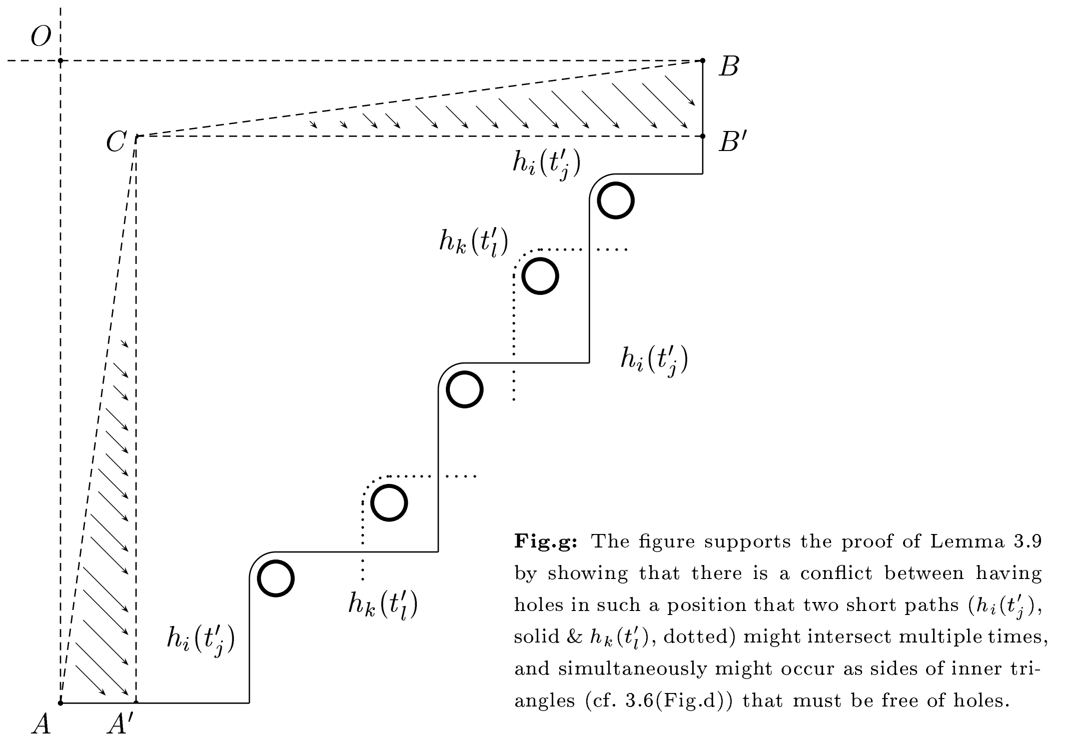


Fig.g: The figure supports the proof of Lemma 3.9 by showing that there is a conflict between having holes in such a position that two short paths ($h_i(t'_j)$, solid & $h_k(t'_l)$, dotted) might intersect multiple times, and simultaneously might occur as sides of inner triangles (cf. 3.6(Fig.d)) that must be free of holes.

be finitely triangulated in such a way that all point $h_i(Q'_j)$ as natural vertices of the triangles $h_i(\delta)$ are vertices of this triangulation.

Proof: By Schönflies' Theorem, each $h_i(\delta)$ bounds a disk that will be called "Disk$(h_i(\delta))$". Denote the union of all those disks by $R \subset \mathbb{R}^2$, and denote by $(P_\nu)_{\nu \in J}$ the system of points consisting of all triangle-vertices $h_i(Q'_j)$ and of all intersection points (including those points where curves turn away or join on, but not the interior points of coinciding segments) between curves $h_i(t'_j)$. The only possible obstruction for a triangulability of R would come from an accumulation of this set of points that need to become vertices of a triangulated R. By the finiteness of the triangulation of V the only chance to have this set infinite would be, if two curves $h_i(t'_j)$ and $h_k(t'_l)$ intersected infinitely many times. We will rule out this possibility in the remainder of this proof:
Since by 3.6(xiii) both curves are monotone, even $\#(h_i(t'_j) \cap h_k(t'_l)) > 3$ implies that these curves must run averagely into the same direction. 3.6(xiii) also implies that these curves, if one of them should have coinciding x- or y-coordinates at its endpoints, must be entirely horizontally or vertically, and therefore cannot be multiple times crossed by another monotone curve. Thus by the left/right-symmetry of the plane (cf. the caption of 3.7(Fig.f))) the situation depicted in Fig.g is already a rather general one. Our curves $h_i(t'_j)$ and $h_k(t'_l)$ belong to inner triangles $h_i(\delta)$ and $h_k(\delta)$, which are by 3.6(xi)/3.7 free of holes. Fig.g shows the situation where both inner triangles lie on the upper left of our curves. This is, indeed, the only possibility that allows multiple intersection, since by 3.6(viii) in the other case our curves, because they cannot get hung up by holes when pushed towards the lower right, would have to consist of one vertical and one horizontal segment, only. Hence we have to assume, as depicted in Fig.g, that both curve have

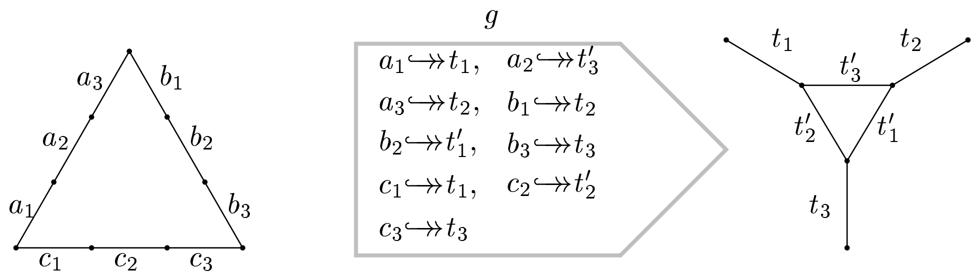


Fig.h (belonging to 4.1(iv)): This figure illustrates the behaviour of the g-factor of some map on some triangle as we usually understand it when we say that the map "factors through $T_3^{\triangle}$".

their non-coinciding system of holes by which they are hung up and brought into their multiply intersecting position, in principle on the same side, so that the holes by which one of the curves is hung up, are an obstacle to complete the other curve to a triangle that is free of holes. Fig.g also shows why this obstacle actually gives a contradiction: Assume that two of the three falling-apart points $h_i(Q_1')$, $h_i(Q_2')$, $h_i(Q_3')$ lie at A and B. If then the third of these points would lie to the upper left of O (which is defined by having the x-coordinate from A and the y-coordinate from B), the entire triangle $h_i(t_j') \cup \overline{B\ O} \cup \overline{O\ A}$ would be as part of $h_i(\delta)$ free of holes, and this would rule out any chance to place holes so that another short multiply intersecting curve $h_k(t_l')$ can lie partially to the upper left of $h_i(t_j')$. However in the other case, the fact that $h_i(t_j')$ is free of holes and the shortening mechanism 3.6(viii) will cause that at least one of the the falling apart points $h_i(Q_j')$ cannot lie at A or B, but, e.g., at A' or B', respectively. Since thus the system of Jordan curves $h_i(\delta)$, each with only three intervals of monotony, can only finitely times intersect, there will by no obstacle to span a system of straight line segments into $R := \bigcup_i(\mathrm{Disk}(h_i(\delta))$ that gives together with segments of the $h_i(\delta)$ curves the desired triangulation for R. $\square$

4. Simplifying a cycle

4.1 Remark: In order to prove our Technical Lemma 1.3, we will make use of the polyhedric pictures of cycles as constructed in Sect.2. For a given map $f : V \to X$ where X is a Sierpińskian Carpet (cf. 3.4), a chain of homotopic maps is to be constructed in this paragraph, each of which has a bit better properties than its predecessors. The property that we finally want to achieve is:

(i) $f' : V \to X$ factors through a graph, i.e. can be decomposed according to $V \xrightarrow{g} \Gamma \xrightarrow{h} X$, where Γ is a graph.

With the proof of this item in 4.7 also our Lemma 1.3 will be proven. As intermediate steps on our way from an arbitrary f to such an f' with $f \simeq f'$, the following properties will also be of importance:

(ii) The restriction of $V \to X$ to the one-skeleton of V factors through a graph (cf. 4.6).

(iii) The restriction of $V \to X$ to the boundary of each triangle $\Delta \in V^{[2]}$ factors through a tree (cf. 4.5).

(iv) The restriction of $V \to X$ to the boundary of each triangle $\Delta \in V^{[2]}$ factors either through a graph T_3^Δ or through one of the degenerations as pictured in 3.7(Fig.e), where the g-factor is described by the labellings in Fig.h. In case the δ-part of T_3^Δ is not degenerated, its image under h is a Jordan curve in $\mathbb{R}^2$ whose interior belongs to X (cf. 4.3).

(v) $V \to X$ takes on different values on all neighbouring vertices of V(cf. 4.2).

4.2 Proposition: *With basic assumptions as in 4.1, for an arbitrary polyhedric picture of a cycle $f_0 : V \to X$, there exists a homotopic map f_1 which satisfies 4.1(v).*

Proof: A Sierpińskian Carpet offers enough local path-connectedness to slide each point by a homotopy a bit apart. This can be done, wherever necessary at a vertex of V, and these homotopies can be extended to the entire complex V. By this we also have a prerequisite so that no short path $f|_e$ where e is an edge of V needs to be constant. □

4.3 Proposition: *Each f_1 with properties from 4.2 is homotopic to some f_2 which for each triangle $\Delta \in V^{[2]}$ in the sense of 4.1(iv) permits a decomposition*

$$f_2|_{\partial\Delta} : \partial\Delta \xrightarrow{g_2^\Delta} T_3^\Delta \xrightarrow{h_2^\Delta} X. \tag{1}$$

Proof : By [Zas5] (cf. 3.6(v)/3.6(x)) every path is relatively homotopic to an algebraically shortest path. f_2 is constructed by applying these homotopies to each edge in V and by extending them to the entire V. By 4.2 this can never give a constant path and [Zas5].E.1.6–7 guarantees a parametrization without constant segments. The remaining claims then follow from Lemma 3.7. Since $\partial\Delta$ is nullhomotopic in V, so must be its image in X. And since by properties of the plane a Jordan curve in $\mathbb{R}^2$ can only be contracted via its interior, the last claim of 4.1(iv) also follows.

4.4 Remark and Definition:
According to 4.1(iv)/3.7 in the case without degenerations each g_2^Δ maps the initial and the end-segment of an edge to one of the t_i-segments of T_3^Δ, and the segment in between to a t'_j-segment. However, please observe that this subdivision of an edge e of V may be very different for different triangles $\Delta \in V^{[2]}$ that have e in its boundary. In the following we therefore will denote as *"outer segments"* of e those parts which neither by any of the g_2^Δ of the adjacent $\Delta \in V^{[2]}$ are mapped to any of the t_i-edges, nor where any of the equivalent segments of e (cf. 3.8) is mapped to a t_i-edge. The complement of the outer segments is called *"inner segment(s)"*. According to the triangulation-construction in 3.9, all endpoints of all inner segments belong to $\{P_\nu \mid \nu \in J\}$, and our triangulation has been constructed in such a way that the restriction of f to each inner segment clearly runs along an edge-path of the triangulation of R. In the forthcoming construction we will not change this property, although we will change the definition of f on segments that we have just defined to be "inner". Observe that our definition of "inner" and "outer" segments is based on f_2 (and therefore is independent of which of

these segments are actually lying in the interior of an edge), and it will not be changed in the forthcoming constructions of this paragraph. This also holds for our understanding of equivalence in the sense of 3.8: We have now used this definition with respect to f_2, and we do not plan to redefine the equivalence, if we change f later.

4.5 Proposition: *Each f_2 with properties from 4.3 is homotopic to some f_3 which satisfies in addition 4.1(iii).*

Proof: In order to reach this goal we redefine f on all inner segments in the sense of Def.4.4. To prepare this definition we choose a one-dimensional spine in our triangulated domain R (cf. 3.9), and extend the spine by adding thorns in such a way that all points P_ν also lie on this extended spine. We then choose a strategy of collapsing R down to its spine, and this strategy shall be our guideline of how to define f_3 out of f_2. A "collapsing strategy" is a concrete order in which we pick a free edge e_0 of some triangle $e_0 \cup e_1 \cup e_2$ of R and replace it by the concatenation of the two other sides $e_1 \cup e_2$. If the restriction of f_2 (or of the current version of f during this process of defining f_3 out of f_2) should run along an edge to be collapsed, the definition of f on the appropriate segment of $V^{[1]}$ is to be changed, and to be replaced by a map that gives for this segments a monotone parametrization of $e_1 \cup e_2$. If f on several inner segments should run along e_0, we replace f on all those segments, and we replace it synchronously, i.e. so that points that before have been mapped to the same point on e_0, are now mapped to the same point on $e_1 \cup e_2$. Therefore it will not be necessary to change the g-factors in the factoring through diagrams that we constructed for each triangle $\Delta \in V^{[2]}$ in 4.3: If some segment that via g_2^Δ goes to a t_i-segment should count as an inner segment and hence should via f be mapped to some edge of the triangulation of R, then, if this segment of f will have to be replaced, it will be replaced on both edges of Δ simultaneously so that not the g-factor but just the h-factor of 4.3(1) needs to be changed. Also observe that none of the changing of the definition of f and of h makes it necessary to change the definition at the endpoints of inner segments, since all these points are mapped to the extended spine of R. That way we can step by step inductively, according to an inductive process that collapses R down to its spine, change the definition of f_2 into a new mapping that we will call f_3. Since each collapsing-process is a homotopy, $f_2|_{V^{[1]}}$ is homotopic to $f_3|_{V^{[1]}}$ and that way the definition of f_3 can be extended to the entire V.

Let us now check that this f_3 satisfies the desired factoring-through properties with respect to an arbitrary triangle $\Delta \in V^{[2]}$. The segments ${g_2^\Delta}^{-1}(\delta)$ have all been inner segments of $V^{[1]}$ and together they have by f_2 been mapped onto a nullhomotopic edge-path of R. Now by f_3, they have to be mapped onto a nullhomotopic curve in the one-dimensional spine of R. Hence this curve lifts to a closed curve inside the universal cover of this spine, which is then a tree. The trace of this lift is a subtree τ, and to this tree τ we accordingly glue edges t_1, t_2 and t_3. We have already stressed that it was not necessary to change the g-factor on ${g_2^\Delta}^{-1}(t_1 \cup t_2 \cup t_3)$, and so we obtain the desired factoring-through-tree τ_Δ by taking T_3^Δ and exchanging δ by τ there. $\square$

4.6 Lemma: *For $f_3 : V^{[1]} \to X$ as constructed in 4.5, also a graph Γ and maps $g_3 : V^{[1]} \to \Gamma$ and $h_3 : \Gamma \to V$ with $f_3 = h_3 \circ g_3$ can be constructed, so that our f_3 also fulfills 4.1(ii).*

Proof: Γ results by identifying the trees τ_Δ that we individually constructed in Step 4.5 for each $\Delta \in V^{[2]}$ separately. The maps which provide this identification have the following origin: If an edge e belongs to two adjacent triangles Δ_1, $\Delta_2 \in V^{[2]}$, then a point in τ_{Δ_1} is identified with a point in τ_{Δ_2}, if according to $g_3^{\Delta_1}$ and $g_3^{\Delta_2}$ they have the same preimage in e. Of course, it needs a transitive closure of these elementary equivalences to obtain Γ. We need to make sure that this transitive closure does not cause dense self-identification of a tree τ_Δ in order to make sure that Γ will actually be a graph. In this context observe that Step 4.3 and all forthcoming steps made sure that all g_3^Δ-maps always provided monotone maps to the the various τ_Δ-trees and the various factoring-through-trees, and any of these segments has by every h-factor been mapped to a nowhere constant path. That way the transitive closure cannot cause self-identifications in a neighbourhood of monotony, so that the result will be a graph. $\square$

4.7 Lemma: *Let $f'|_{V^{[1]}} := f_3|_{V^{[1]}}$ with f_3 as constructed in 4.5. This map can be extended to the entire V so that the resulting map, that we will denote by f', is homotopic to f_3 while factoring through the same graph Γ that we constructed in 4.6 for $f_3|_{V^{[1]}}$. Hence f' satisfies 4.1(i).*

Proof: The definition of the remaining parts of f' will be by an inductive process skeletonwise, where in 4.6 the zero-step has been completed with the definition on the one-skeleton. Since a skeletonwise definition in particular means that, when defining f' on the k-skeleton, all given definitions on lower-dimensional skeletons will be respected, the actual definition will be simplexwise.
Let Δ be a k-simplex of V, and assume that on $\partial\Delta$ in the course of our inductive process already some version of a map of f is given, which satisfies to factor through Γ. The map f_3, as constructed in 4.5, may be interpreted as some version of our map f that is given in the interior of our simplex, so that this map is homotopic to the given map f_3, but in the interior it does not satisfy to factor through Γ. For the boundary, on the other hand, we have: $f_3|_{\partial\Delta} : \partial\Delta \xrightarrow{g_3|_{\partial\Delta}} \Gamma \xrightarrow{h_3} X$. Our new f, i.e. the appropriate part of f', shall in the interior of Δ be defined by extending $g_{3|\partial\Delta}$ to the interior of Δ and concatenating it with h_3. An appropriate extension of $g_{3|\partial\Delta} \to \Gamma$ exists, since Γ as a graph is aspherical. Only for $k = 2$ with $\dim(\partial\Delta) = 1$ this argument is not yet valid, but for this dimension in 4.5 we made sure that that here the factoring is actually through a tree. Having that way by extending g_3 defined our desired new map f', it follows directly from the asphericity of X (cf. [Zas5].(Thm.A)/[CCZ]) that this f' is homotopic to f_3 rel. $\partial\Delta$. This completes the inductive construction of f'. $\square$

4.8 Proof of Theorem 1.2:
Let $Z \subset \mathbb{R}^2$ and $f : V \to Z$ be a polyhedric picture of a k-cycle with $k = \dim(V) \geq 2$. By 1.3/4.7 a homotopic version of f can be decomposed according to $f \simeq f' = h \circ g$ with $g : V \to \Gamma$ and Γ being one-dimensional. Since by [C–F3].(Thm.5) or [E–K3].(Thm.2.1)

the higher singular homology of Γ vanishes and since g can be interpreted as a polyhedric picture for a k-cycle of Γ, g can be extended to a map $G : W \to \Gamma$, where W is a $(k+1)$-dimensional polyhedron with $\partial W = V$. Thus G describes the bounding chain in $C_{k+1}(\Gamma)$ for the cycle that was described by g, and $h \circ G$ describes the appropriate bounding chain for f'. Since the homotopy $f \simeq f'$ induces an algebraic homotopy on chain level, we have that $f = 0$ in $H_k(Z)$. □

Observe that above in 4.8 it was actually not necessary to quote the [C–F3]- or [E–K3]-paper from wild algebraic topology, since the quotient space as constructed there was a simplicial graph, i.e. tame. Hence the above quote could have also been to elementary results from simplicial topology and Eilenberg-Steenrod coincidence theory ([E–S].III.10.1/VII.§10). However, none of these possible references is elementary, and this is why we point out in 5.6 how with the same type of arguments as above the proof could also have been completed without external quotes.

5. A geometric interpretation of Milnor's and Barratt's example.

The purpose of this section is to contrast the preceding result for planar sets with a classical result of the opposite conclusion for a subset of $\mathbb{R}^3$ (cf. 1.1). We therefore first recall this result and describe it in the language that we developed for obtaining our main result Thm.1.2/4.8. With the final remarks of this section we try to indicate on the technical level why planarity was essential in order to achieve a proof for 1.2.

5.1 Definition: *"Harmonic Balls"* we call a topological space Y^2 which is obtained by embedding countably many round two-spheres into $\mathbb{R}^3$ in such a way that they all have precisely one point in common and that their radii are given by a null-sequence. *"Hawaiian Earrings"* Y^1 are analogously composed by infinitely many round circles and defined by a similar embedding into $\mathbb{R}^2$ (cf. 1.6(Fig.a)).
Although two-dimensional, the Harmonic Balls have a non-trivial singular H_3. In our attempt to geometrically interpret this result of [M–B], we construct in the following a cycle by its polyhedric picture and prove that it is not nullhomologous. This construction is based on an infinite plugging of Hopf-maps.

5.2 Remark: Recall that the classical Hopf-map ([Hopf1], [Hopf2]) can be geometrically interpreted as follows: Represent S^3 as $V_1 \cup V_2$, where V_1 and V_2 are both solid tori that are glued together along their boundary, so that the meridian of V_1 is identified with the longitude of V_2 and vice versa. That way, when both solid tori are locally trivially Seifert-fibred so that each S^1-fibre is homotopically the sum of one meridian and one longitude, this fibration continuously extends from V_1 to V_2. Observe that this gives a Seifert-fibration of S^3 without exceptional fibres, since also the core-curves of V_1 and V_2 continuously fit into this fibration.
When quotienting each of these fibres to one point, we obtain S^2. The quotient map $S^3 \to S^2$ is the classical Hopf-map, and each of the above described fibres is a level-set.

5.3 Definition of the modified Hopf-map:
With respect to the picture of 5.2, thicken one of the fibres to a solid torus V_3. Regard this thickening-process as a homotopy, which leaves the other fibres unchanged, apart from, that the fibres in a neighbourhood of V_3 have been pushed a bit aside. The quotient map, which factors each ordinary fibre to one point, but also factors the entire V_3 just to one point is homotopic to the classical Hopf-map. We regard this map as "modified Hopf-map", which is homotopic to the classical one. The complement of V_3 we regard as the *"support" of the map.* Having changed the classical Hopf-map in such a way that it is just supported on a solid torus was the point of this construction.

5.4 Definition: The purpose of this construction is to define a map $V^3 = S^3 \xrightarrow{f} Y^2$ that will be the polyhedric picture of a nonzero element in $H_3(Y^2)$ (cf. 5.1):
Choose a flat round disk D^2 in S^3. Inside D^2, starting with its boundary curve choose a countable sequence of concentric disjoint circles that accumulate only towards the midpoint of D^2. Denumerate these circles monotonically by the naturals starting with "one" for ∂D^2, "two" for the nextinner circle, and so on. Denote the i^{th} of these circles by k_i, and span them by mutually disjoint disks D_i, which may be constructed as caps over D. Associate with all of the k_i-circles disjoint regular neighbourhoods in S^3, which then are all solid tori and analogously denumerated. Finally define a map $S^3 \to Y^2$ by mapping the complement of these tori to P and the i^{th} circle on D^2 to P_i (cf. 1.6(Fig.a)), and interpolate between these parts by using a modified Hopf-map into the i^{th} two-sphere of Y^2 (more precisely: the restriction of this Hopf-map to its support, cf. 5.3) for defining our map on the complement of the core curve of the i^{th} solid torus. Since that way the infinitely many solid tori in a neighbourhood of the mid-point of D^2 map only to the small spheres of Y^2, and since these spheres with huge indices form a neighbourhood of P according to the topology of Y^2, this construction gives a continuous map. Finally, since S^3 is a closed manifold-polyhedron, such a map defines a cycle c via its polyhedric picture (cf. Setc.2).

5.5 Proposition: *c as defined in Def.5.4 is is not nullhomologous in $H^3(Y^2)$.*

The Proof will be a proof by contradiction, i.e. we will assume that $F : W^4 \to Y^2$ is the polyhedric picture for a bounding chain for c, but then show that these demands cannot be satisfied by a polyhedron. However, at first let us assume they can. $\partial W^4 = S^3$ and, without loss of generality, a neighbourhood of the boundary component is of type $S^3 \times I$. Let $W' := W^4 - (S^3 \times I)$ and note that only the one- and zero-skeleton of W' contain non-manifold points of this almost-manifold-polyhedron W^4. By general position, $F^{-1}(P_i)$ (cf. 1.6(Fig.a)) avoids the one-skeleton in W' and approaches the boundary as a vertical surface, i.e. $F^{-1}(P_i) \cap (S^3 \times I) = k_i \times [0, 1]$. Here and in the remainder of this proof all Cartesian product-decomposition refer to canonical decomposition of the neighbourhood of ∂W^4 into an S^3-component and a "height parameter" $\in I = [0, 1]$ with $S^3 \times \{0\} = \partial W^4$ and $S^3 \times \{1\} = \partial W'$. We choose two systems of disjoint and mutually disjoint height-parameters $\{\kappa_i, \kappa_i' \mid i \in \mathbb{N}\}$ and let $U_i := F^{-1}(P_i) - (k_i \times [0, \kappa_i)) \cup (D_i \times \{\kappa_i\}$, where D_i has been chosen in 5.4. By using the other system of height-parameters κ_i' and with $P_i' \neq P_i$ chosen on the i^{th} sphere of Y^2, we construct "parallel copies" of these surfaces U_i according to $U_i' := F^{-1}(P_i') - (k_i' \times [0, \kappa_i')) \cup (D_i' \times \{\kappa_i'\})$. In general, the prime ("$'$") in this proof is used to denote these parallel copies. By construction, we have that $U_i \cap U_j = \emptyset$ for $i \neq j$, but that $\#(U_i \cap U_i') = 1$. This result is a consequence of the facts that in the W'-part the surfaces are constructed as complete preimages of different points and therefore disjoint, and the same for $i \neq j$ in the $(S^3 \times I)$-part, since k_i and k_j are unlinked and hence have disjoint caps D_i and D_j; but since as property of the Hopf-map our "parallel circle" k_i' must be Hopf-linked with k_i, a cap D_i' cannot be disjoint from t_i and vice versa. Hence, depending on whether $\kappa_i' < \kappa_i$ or vice versa, either $U_i \cap U_i'$ has an intersection point at $(D_i \cap k_i' , \kappa_i)$ or at $(D_i' \cap k_i , \kappa_i')$, respectively. The closed surfaces U_i lie in W^4, but with N denoting the regular neighbourhood also in the compact four-manifold $W^4 - N({W'}^{[1]})$. By our geometrical analysis of the intersectional behaviour, the intersection form $\langle \cdot, \cdot \rangle$ of this compact four-manifold tells us that $\langle [U_i], [U_j] \rangle = \pm\delta_{i,j}$ with $\delta_{i,j}$ being the Kronecker-symbol. I.e. the $U_i \simeq U_i'$ would represent infinitely many linearly independent vectors in $H_2(W^4 - N({W'}^{[1]}), \mathbb{R})$ but so many linearly independent vectors cannot exist in a second homology group of the triangulable space $W^4 - N({W'}^{[1]})$. This contradiction shows that c cannot possess a bounding chain in singular homology. $\square$

We may interpret the above arguments as a geometric version of the [M–B]-result.

5.6 Penultimate Remark: In preliminary drafts of this paper there was more relation visible between Sect.5 and the preceding paragraphs. Namely, the author started his investigations in this area by asking whether independently of all axiomatic coincidence theory for homology groups (which usually does not apply to non-triangulable spaces) there is a direct argument from singular homology theory why usually spaces should not have singular homology groups for indices exceeding the dimension of the space. In generalizing the example from 2.2(Fig.b), the author came upon a method for how one could always try to construct a candidate for a bounding chain for the appropriate dimensions, if one has a cycle as given by its polyhedric picture $f : V^k \to X$. Namely, if one thinks of V^k decomposed according to the level-surfaces $f^{-1}(\{x\})$, one might always try to replace each of these level-surfaces (or more precisely: its path-components) by a cone over these path-components and that way try to turn our k-dimensional space into a $(k+1)$-dimensional space ("fibre-wise cone"). The constant value of the map f on each level-surface can be extended to the entire cone, and so the entire map continuously to the space to be constructed. Usually the so constructed space V^{k+1} should fit into the framework as described in 2.1. However, the [M–B]-example (cf. Thm.1.1/5.4–5) shows that this construction method will not always work, and the author's conclusion was, that the critical point is, whether the resulting space will be again triangulable. For too wild f it will not be, and indeed, the proof that we completed in 5.5 can be interpreted as a proof that rules out triangulability of V^{k+1} — regardless, whether it is constructed by this or by another method.

However, for planar sets it is always possible to achieve triangulability by extending the taming-process that has been described in Sect.4 by a few additional steps. Recall that the last step of the proof of Lemma 4.7 allowed us to replace in the interior of a simplex our map by any continuous map that we like — as long, as we did not change

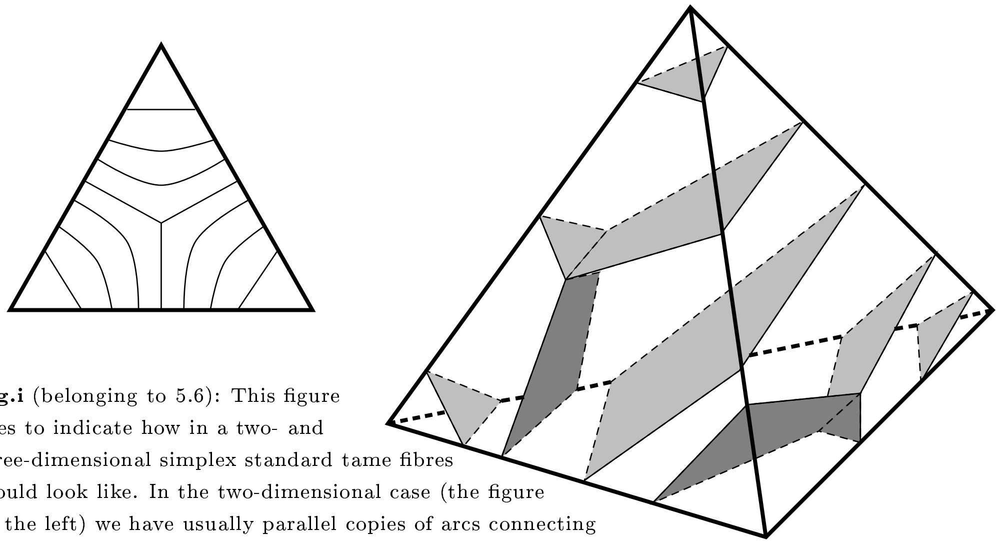


Fig.i (belonging to 5.6): This figure tries to indicate how in a two- and three-dimensional simplex standard tame fibres should look like. In the two-dimensional case (the figure on the left) we have usually parallel copies of arcs connecting two opposed edges, and one T_3-graph as "branching fibre" in the middle. In the three-dimensional case (the figure on the right) we have parallel copies of normal surfaces and two branching fibres that separate the regions of parallel copies of these normal surfaces.

the map on the boundary. Hence in particular, if we think of a triangle whose map on the boundary factors through a T_3-tree (cf. 3.7(Fig.e)), we can that way replace f in the interior, e.g. by the map that has such tame level-surfaces as we pictured in the two subfigures of Fig.i. It is for any dimension possible to make such constructions, since it just requires a straightforward proof by induction to see that a map on the one-skeleton of higher-dimensional simplices, if the boundary of every two-face factors through a T_3-tree, has at least to factor through some tree. Additional technical problems (e.g. to reduce the situation from the trees that we actually constructed in Sect.4 (cf. esp. 4.5/4.6) to T_3-trees (cf. 3.7(Fig.e)) and to discuss possible degenerations of the pictures in Fig.i) can be overcome. More precisely, degenerations can actually be avoided by inserting "artificial thorns" into our maps (cf. Fig.j). That way the proof of Theorem 1.2 could have also been completed elementarily by continuing the taming-process and constructing the bounding chain directly via the fibre-wise cone-constructions, since, when all fibres look as tame as in Fig.i, this construction would definitely end with a triangulable space. However, since the proof of Theorem 1.2 was completed in 4.8 by appropriate quotes, it does not seem worth to carry out all this lengthy constructions elementarily, also.

5.7 Concluding Remark: Recall that the non-nullhomologous cycle c of 5.4–5.5 was based on Hopf-maps. Any Hopf-map is nullhomologous, but because it is the generator of $\pi_3(S^2)$ it follows that its bounding chain cannot be modelled on B^4 — it needs some complication in topology to construct a bounding chain for the Hopf-map. Hence the scheme of the above proof was to show that it takes "infinite complication in topology" to construct a bounding chain for our infinite plugging of Hopf-maps, and such "infinite complication in topology" can never be given by a finite polyhedron. In the above technical proof the consideration of intersection and linking numbers was our

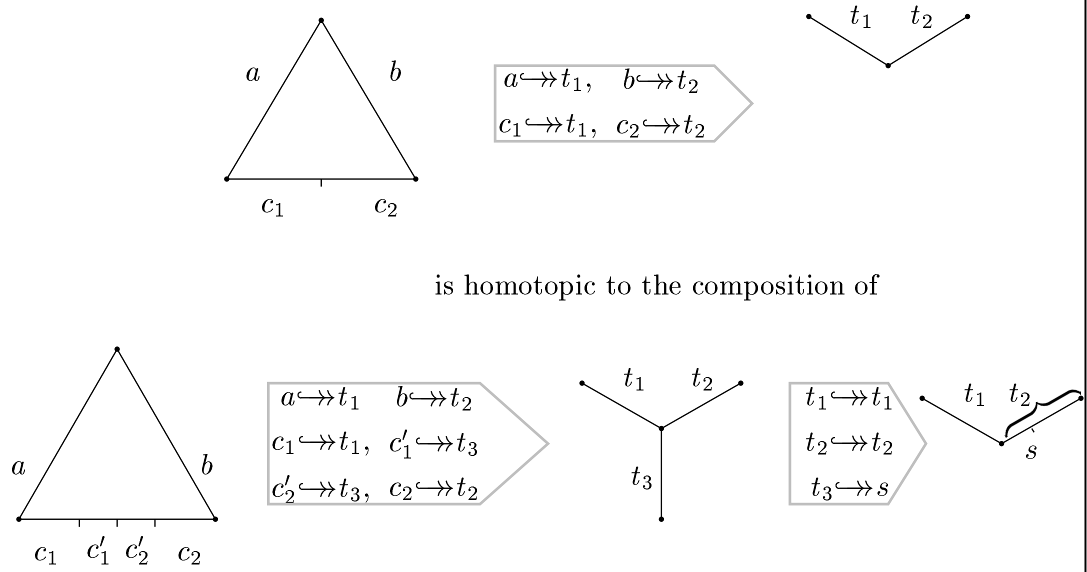


Fig.j (belonging to 5.6): Degenerations with respect to the picture Fig.i may occur if k-dimensional domains that are in the standard picture fibred by parallel non-branching fibres collapse and either the bounding branching fibres have to be united to give more complicated branched fibres, or the branching fibres are not in a strictly dual position to the skeleton of V any longer. However, this picture shows how technically such degenerations can be avoided by just homotopying f to be a bit more complicated.

measurement for "complication in topology". One might also interpret (cf. the final proof-argument of 5.5) the result in that way that the phenomenon of a non-trivial H_3 for a wild two-dimensional complex in [M–B]'s example is originated by the fact that all chains in singular homology theory must be finite — in some analogy to the phenomenon of $\pi_1(Y^1)$ being not free, cf. Remark 1.6 in the introduction.

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Andreas Zastrow
Instytut Matematyki
Uniwersytet Gdański
ul. Wita Stwosza 57
80-952 Gdańsk
POLAND
Email: zastrow@math.univ.gda.pl